Supersymmetric solutions with SU(3), G2 and Spin(7) structures

（SU(3),G2 及び Spin(7)構造を持つ超対称性解）

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Preface

This thesis is based on the following three papers:

1. "Supersymmetric heterotic solutions via non-$SU(3)$ standard embedding" [1],

2. "Heterotic solutions with G_2 and $Spin(7)$ structures" [2],

3. "General Wahlquist solutions in all dimensions" [3].

The main bodies of this thesis are based on the papers 1(Chapter 4) and 2(Chapter 3), which are described on constructing supersymmetric solutions with $SU(3)$-, G_2-, and $Spin(7)$-structures in heterotic supergravity theory. Appendix F is based on the paper 3 in the view of torsion geometry, which is described on Killing-Yano symmetry of the Wahlquist spacetime and generalizing the Wahlquist metric to higher dimension in General relativity.

Abstract

In this thesis, we study supersymmetric solutions in 6-, 7-, and 8-dimensional supergravity theories, where an $SU(3)$-structure on 6-dimensional manifolds, a G_2-structure on 7-dimensional manifolds, and an $Spin(7)$-structure on 8-dimensional manifolds play an important role. In dimension 6, we obtain an intersecting metric by superposing two Gibbons-Hawking metrics with conformal factors. Then, the metric, the fundamental 2-form and the complex $(3,0)$-form satisfy the defining equations of the $SU(3)$-structure associated with type II supergravity whose geometry consists of an exact Lee form and a closed Bismut torsion, which are identified with the dilaton and the 3-form flux respectively. It is shown that the corresponding six dimensional manifold is a Calabi–Yau manifold with torsion describing a supersymmetric solution of the theory. By using $U(1)$ isometries the solution is twice T-dualized, which leads to the supersymmetric solution with $SO(6)$ holonomy. In dimension 7, a G_2-structure associated with Abelian heterotic supergravity theory is introduced. Since the torsion 3-form is closed and the Lee form is exact, they are identified with the 3-form flux and the dilaton as in the case of the $SU(3)$-structure. The G_2-structure is determined by the fundamental 3-form, it's Hodge dual 4-form and the field strength of $U(1)$ gauge field. For the 7-dimensional manifold, we assume the cohomogeneity one manifold of the form $\boldsymbol{R}_+ \times S^3 \times S^3$. We obtain 2 types of regular supersymmetric solutions, i.e., S^3-bolt and $T^{1,1}$-bolt solutions. In dimension 8, a $Spin(7)$-structure associated with Abelian heterotic supergravity theory is introduced. Then, the $Spin(7)$-structure is determined by a fundamental 4-form and a field strength of $U(1)$ gauge field. Using an ansatz of 3-Sasakian manifolds, we construct supersymmetric regular solutions.

Contents

Chapter 1

Introduction

String theory has been thought of as an important candidate of theories that unify gravity and the other fundamental forces. One of the most exciting predictions is that our world might have extra dimensions because the theory is required to be defined in ten dimensions. However, as our "visible" spacetime is in four dimensions, one needs to explain why we see the four-dimensional spacetime. One possible solution is the idea of compactification, where ten-dimensional spacetime M_{10} is given by a product of our four-dimensional spacetime $\boldsymbol{R}^{1,3}$ and a compactified six-dimensional space M_6, $M_{10} = \boldsymbol{R}^{1,3} \times M_6$. The "invisible" six-dimensional space is called an internal space.

In superstring theory, one requires invariance of the theory under supersymmetry transformation in ten dimensions, which does not necessarily require invariance in four dimensions. The presence of supersymmetry is helpful in solving equations of motion with a special geometric structure of the internal space. In heterotic string theory, for example, necessary and sufficient conditions for preserving supersymmetry in $\boldsymbol{R}^{1,3}$ are equivalent to the vanishing of supersymmetry variations for the gravitino ψ, the dilatino λ, and the gaugino χ on the internal space M_6,

$$\delta\psi \propto \left(\nabla_\mu + \frac{1}{8} H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right)\varepsilon = 0\,, \tag{1.1}$$

$$\delta\chi \propto \left((\partial_\mu\varphi)\gamma^\mu + \frac{1}{12} H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right)\varepsilon = 0\,, \tag{1.2}$$

$$\delta\lambda \propto F_{\mu\nu}\gamma^{\mu\nu}\varepsilon = 0\,, \tag{1.3}$$

where $H_{\mu\nu\rho}$, φ, and $F_{\mu\nu}$ denotes the 3-form flux, the dilaton, and the field strength of the gauge field, respectively. The parameter of supersymmetry transformations, denoted by ε, is called a Killing spinor on M_6. Once at least one Killing spinor exists, the internal space M_6 admits a special geometric structure, called a G-structure, and the holonomy group shrinks to a group G that is characterized by special differential forms, called G-invariant forms or fundamental forms, associated with the G-structure. Thus, supersymmetry conditions that are expressed in terms of Killing spinors are encoded to those in terms of the G-invariant forms, which provides us one way to study supersymmetric solutions, specifically, of the Neveu–Schwarz (NS) sector in heterotic string theory and of the NS-NS sector in type II string theory [4, 5].

In superstring theory, one often considers the situation in which the matter fields are absent ($H_{\mu\nu\rho} = \varphi = F_{\mu\nu} = 0$) because the supersymmetry variation equation for the gravitino simplifies to a parallel spinor equation of the Levi-Civitá connection, and a Ricci-flat Kähler manifold, known as a Calabi–Yau manifold, is chosen as the internal space M_6. In complex geometry it is known that holonomy of a Kähler manifold is contained in $U(n)$. In particular, if a Kähler manifold admits a Ricci-flat metric, the holonomy shrinks to $SU(n)$. It is also known that a Kähler manifold with $SU(n)$ holonomy has a parallel spinor. Conversely, if a parallel spinor exists on a Kähler manifold, the Kähler manifold admits the $SU(n)$ holonomy. Furthermore, the Calabi conjecture and its proof by Yau states that the Kähler manifold admits a unique Ricci-flat Kähler metric if the first Chern class vanishes [6, 7].

$U(n)$ and $SU(n)$ holonomy groups on Riemannian manifolds are examples of the special holonomy groups classified by Berger [8] (see also [6]). If there exists at least one parallel spinor on an n-dimensional Riemannian manifold, the holonomy groups and the number of independent parallel spinors are provided by the following table [6] (Theorem. 3.6.1),

Name	dimension n	$Hol(\nabla)$	# of spinor (N_+, N_-)
Calabi–Yau	$4m$	$SU(2m)$	(2,0)
Calabi–Yau	$4m+2$	$SU(2m+1)$	(1,1)
Hyper Kähler	$4m$	$Sp(m)$	(m+1,0)
Parallel G_2	7	G_2	1
Parallel $Spin(7)$	8	$Spin(7)$	(1,0)

where $N_\pm$ denotes the number of positive or negative spinors.In particular, every Riemannian manifold with one of special holonomy groups other than $U(n)$ admits at least one parallel spinor. Metrics on such manifolds are Ricci-flat metrics. Ricci-flat metrics on a Riemannian manifold with a special holonomy group have been investigated actively. If there exist G-invariant forms of a G-structure on an n-dimensional Riemannian manifold, the holonomy group is contained in G. The corresponding Ricci-flat metrics are constructed by the G-invariant forms, which are determined by first-order differential equations of the G-invariant forms. Consequently, a question of construction of a Ricci-flat metric on a Riemannian manifold admitting a special holonomy group arrives at a problem with solving the first-order differential equations of G-invariant forms.

G-structures can be also applied to constructing not only Ricci-flat metrics but also solutions of supergravity theories. A G-structure related to special holonomy is classified into some classes by first-order differential equations of the corresponding G-invariant forms. Some specific classes give necessary conditions for supersymmetry preservation in supergravity theories [9]. Ricci-flat metrics are obtained when the first-order differential equations of G-invariant forms takes the most simple form. In other class, the equations of G-invariant forms give rise to a gravitational part of equations of motion in supergravity theories. In Refs. [10, 11], it was shown that equations of motion in supergravity theory can be solved with the G-invariant forms if equations of motion for the dilaton and Bianchi identity hold.

An $SU(n)$-structure on a $2n$-dimensional almost Hermitian manifold (M_{2n}, J) is determined by a fundamental 2-form, κ, and a complex $(n, 0)$-form, Υ. If and only if a 2-form κ and a $(n, 0)$-form Υ with a volume-matching condition satisfy

$$d\kappa = 0\ ,\ d\Upsilon = 0\,, \tag{1.4}$$

the manifold, $(M_{2n}, J, \kappa, \Upsilon)$, is said to be a Calabi–Yau manifold. A Calabi–Yau manifold is also a Kähler manifold with $\nabla\Upsilon = 0$, and the holonomy is contained in $SU(n)$. The associated metric is Ricci-flat. The many explicit examples of Calabi–Yau manifolds have been constructed as hypersurfaces in complex projective spaces[6]. In the case where $n = 2$, a famous example of a Calabi–Yau 2-fold is the $K3$ surface, which is a hypersurface in 3-dimensional complex projective space $\boldsymbol{CP}^3$. A $K3$ surface is a compact manifold that has a self-dual metric with self-dual Riemannian curvature. In the case of $n = 3$, an example of a Calabi-Yau metric is the resolved cone metric from the Sasaki Einstein metric [12].

When one considers G_2-structure, it is characterised by the G_2-invariant 3-form Ω satisfying the condition

$$d\Omega = 0\ ,\ d * \Omega = 0\,. \tag{1.5}$$

The corresponding G_2 metric is a Ricci-flat metric with G_2 holonomy [13], which is obtained by solving the first-order equations (1.5). G_2 metrics for the principal orbits $\boldsymbol{CP}(3)$ or $S^3 \times S^3$ have been studied in [5, 14, 15, 16, 17]. Under a cohomogeneity one ansatz, equations (1.5) reduce to ordinary differential equations. These equations were explicitly solved with asymptotically conical (AC) [14, 15] and asymptotically locally conical (ALC) [5, 16, 17] metrics. Similarly, $Spin(7)$-structure is given by the condition

$$d\Psi = 0\,, \tag{1.6}$$

where Ψ is a $Spin(7)$-invariant 4-form satisfying $*\Psi = \Psi$ [18]. The corresponding metrics with $Spin(7)$ holonomy were obtained in the similar manner to G_2 metrics. For $Sp(2)/Sp(1)$ or $SU(3)/U(1)$ principal orbits, $Spin(7)$ metrics have been studied in [16, 17, 19, 20, 21, 22]. Examples of compact G_2- and $Spin(7)$-manifolds were constructed by Joyce [6].

In this thesis, we study supersymmetric solutions in 6-, 7-, and 8-dimensional supergravity theories, where an $SU(3)$-structure on 6-dimensional manifolds, a G_2-structure on 7-dimensional manifolds, and a $Spin(7)$-structure on 8-dimensional manifolds play important roles. In dimension 6, we construct an intersecting metric g_6 by superposing two Gibbons–Hawking metrics with conformal factors. The metric g_6, the fundamental 2-form κ, and the complex $(3, 0)$-form Υ satisfy the defining equations of the $SU(3)$-structure associated with type II supergravity, where the Lee form $\Theta_6 = 2d\varphi_6$ is an exact 1-form and the Bismut torsion H_6 is a closed 3-form. We show that the manifold $(M_6, g_6, \kappa, \Upsilon)$ is a Calabi–Yau with torsion manifold and thus (g_6, H_6, φ_6) are supersymmetric solutions in the theory [11]. In dimension 7, we introduce a class of G_2-structures associated with Abelian heterotic supergravity theory together with a class having a closed 3-form torsion H_7 and an exact Lee form $\Theta_7 = 2d\varphi_7$. This class is determined by the relevant (g_7, Ω, F_7), where g_7 is the metric associated with the fundamental 3-form Ω and F_7 is the field strength of $U(1)$ gauge field satisfying Bianchi identity. Then, we solve the defining equations of the class on cohomogeneity one manifold of the form $\boldsymbol{R}_+ \times S^3 \times S^3$. The quartet $(g_7, \varphi_7, H_7, F_7)$ solves the equations of motion automatically in the theory [23]. In dimension 8, similar to the case in dimension 7, we introduce a class of $Spin(7)$-structure associated with Abelian heterotic supergravity theory. The class is determined by the proper (g_8, Ψ, F_8), where g_8 is the metric associated with the fundamental 4-form Ψ and F_8 is the field strength of $U(1)$ gauge field. We assume the form of manifolds $\boldsymbol{R}_+ \times M_{3-Sasaki}$, where $M_{3-Sasaki}$ is a manifold with 3-Sasakian structure. Then, we construct supersymmetric solutions in the theory.

The remainder of this thesis is organized as follows. In chapter 2, we briefly review the $SU(3)$-, HKT(Hyper Kähler with torsion), G_2-, and $Spin(7)$-structure. In chapter 3, we construct supersymmetric solutions in 7-, and 8- dimensional Abelian heterotic supergravity theory by solving the defining equations of the specific classes. In chapter 4, we construct the supersymmetric solution in heterotic supergravity theory with $SU(3)$ holonomy of the Bismut connection and $SO(6)$ holonomy of the Hull connection.

Chapter 2

G-structures

We review a general method of characterizing a geometric G-structure admitting a G-connection with a 3-form torsion in accordance with Ref. [24]. Let (M_n, g) be an oriented Riemannian manifold and denoted by $\mathcal{L}(M_n)$ its frame bundle. The Levi-Civita connection on the frame bundle $\mathcal{L}(M_n)$ is a 1-form

$$\mathcal{Z}^{LC} \; : \; T(\mathcal{L}(M_n)) \longrightarrow \mathfrak{so}(n) \tag{2.1}$$

with values in the Lie algebra $\mathfrak{so}(n)$. We fix a closed subgroup G of the special orthogonal group $SO(n)$, where G's action is simply transitive.A G-structure on M_n is a G-subbundle $\mathcal{Q} \subset \mathcal{L}(M_n)$. We decompose the Lie algebra $\mathfrak{so}(n)$ into the subalgebra $\mathfrak{g}$ and its orthogonal complement $\mathfrak{m}$,

$$\mathfrak{so}(n) = \mathfrak{g} \oplus \mathfrak{m} \,, \tag{2.2}$$

where $\mathfrak{g}$ is the Lie algebra to Lie group G. Restricting $\mathcal{Z}^{LC}$ to the tangent bundle $T(\mathcal{Q})$ brings about the decomposition of $\mathcal{Z}^{LC}|_{T(\mathcal{Q})}$ corresponding to (2.2),

$$\mathcal{Z}^{LC}|_{T(\mathcal{Q})} = \tilde{\mathcal{Z}} \oplus \mathcal{T} \,. \tag{2.3}$$

$\tilde{\mathcal{Z}}$ is a connection 1-form field on $\mathcal{Q}$ with values in the associated bundle $\mathcal{Q} \times_{Ad} \mathfrak{g}$,

$$\tilde{\mathcal{Z}} \; : \; T(\mathcal{Q}) \longrightarrow \mathcal{Q} \times_{Ad} \mathfrak{g} \,. \tag{2.4}$$

$\mathcal{T}$ is 1-form field on $\mathcal{Q}$ with values in the associated bundle $(\mathcal{Q} \times \mathfrak{m})/G$,

$$\mathcal{T} \; : \; T(\mathcal{Q}) \longrightarrow (\mathcal{Q} \times \mathfrak{m})/G \tag{2.5}$$

and this is known as an intrinsic torsion.

[supplementation]
$\mathcal{Q} \times_{Ad} \mathfrak{g}$ is the vector bundle associated with G-structure $\mathcal{Q}$ constructed by adjoint representation and its fiber is the Lie algebra $\mathfrak{g}$. $\mathcal{Q} \times_{Ad} \mathfrak{g} \equiv (\mathcal{Q} \times \mathfrak{g})/G$ means

$$s \; : \; (u, A) \longmapsto (us, s^{-1}As) \;\; (s \in G \;, \; (u, A) \in \mathcal{Q} \times \mathfrak{g})$$

(u, A) is identified with $(u, s^{-1}As)$.
$\mathcal{Q} \times_\rho \mathfrak{m}$ is the vector bundle associated with G-structure $\mathcal{Q}$ constructed by a representation ρ and its fiber is $\mathfrak{m}$. $\mathcal{Q} \times_\rho \mathfrak{m} \equiv (\mathcal{Q} \times \mathfrak{m})/G$ means

$$s \ : \ (u, y) \longmapsto (us, \rho(s)^{-1}y) \ \ (s \in G \ , \ (u, y) \in \mathcal{Q} \times \mathfrak{m})$$

(u, y) is identified with $(u, \rho(s)^{-1}y)$.

An arbitrary G-connection $\mathcal{Z}$ on $\mathcal{Q}$ differs from $\tilde{\mathcal{Z}}$ by an 1-form Σ with values in the associated bundle $\mathcal{Q} \times_{Ad} \mathfrak{g}$,

$$\mathcal{Z} = \tilde{\mathcal{Z}} + \Sigma = \mathcal{Z}^{LC}|_{T(\mathcal{Q})} - \mathcal{T} + \Sigma\,. \tag{2.6}$$

From now on, we consider the corresponding connection ∇^T to (2.6) on the associated bundle $\mathcal{Q} \times_G \boldsymbol{R}^n$.Hence the connection 1-form field ω^T of ∇^T associated with $\mathcal{Z}$ belongs to $T^*(M_n) \otimes (\mathcal{Q} \times_{Ad} \mathfrak{g})$,

$$\omega^T \in T^*(M_n) \otimes (\mathcal{Q} \times_{Ad} \mathfrak{g}) \cong (\mathcal{Q} \times_G \boldsymbol{R}^n) \otimes (\mathcal{Q} \times_{Ad} \mathfrak{g}) \quad (\because T^*_p M_n \cong \boldsymbol{R}^n)\,. \tag{2.7}$$

On the associated bundle $\mathcal{Q} \times_G \boldsymbol{R}^n$ the intrinsic torsion $\mathcal{T}$ on M_n belongs to $T^*(M_n) \otimes (\mathcal{Q} \times_G \mathfrak{m})$,

$$\mathcal{T} \in T^*(M_n) \otimes (\mathcal{Q} \times_G \mathfrak{m}) \cong (\mathcal{Q} \times_G \boldsymbol{R}^n) \otimes (\mathcal{Q} \times_G \mathfrak{m}) \quad (\because T^*_p M_n \cong \boldsymbol{R}^n)\,. \tag{2.8}$$

At local ω^T and $\mathcal{T}$ belong to $\boldsymbol{R}^n \otimes \mathfrak{g}$ and $\boldsymbol{R}^n \otimes \mathfrak{m}$ respectively:

$$\omega^T(p) \in \boldsymbol{R}^n \otimes \mathfrak{g} \quad (p \in M_n)\,, \tag{2.9}$$

$$\mathcal{T}(p) \in \boldsymbol{R}^n \otimes \mathfrak{m} \quad (p \in M_n)\,. \tag{2.10}$$

We identify 1-forms $\omega^T(p)$ and $\mathcal{T}(p)$ with 1-form fields ω^T and $\mathcal{T}$ respectively. The corresponding connection is given by

$$\nabla^T_X Y = \nabla^{LC}_X Y - \mathcal{T}(X)(Y) + \Sigma(X)(Y) \ \ (X, Y \in E_p(\mathcal{Q} \times_G \boldsymbol{R}^n) = \boldsymbol{R}^n \cong T_p(M_n))\,, \tag{2.11}$$

where ∇^{LC} is the Levi-Civita connection on the associated bundle $\mathcal{Q} \times_G \boldsymbol{R}^n$.Of course, $\nabla^{LC}_X Y \in \boldsymbol{R}^n \otimes \mathfrak{so}(n)$, $\mathcal{T}(X)(Y) \in \boldsymbol{R}^n \otimes \mathfrak{m}$, $\Sigma(X)(Y) \in \boldsymbol{R}^n \otimes \mathfrak{g}$ and $\nabla^T_X Y \in \boldsymbol{R}^n \otimes \mathfrak{so}(n)$. The torsion tensor of affine connection ∇^T is defined by

$$T(X, Y) = \frac{1}{2}(\nabla^T_X Y - \nabla^T_Y X - [X, Y]) \tag{2.12}$$

and then by using (2.11) this is rewritten as follows:

$$\begin{aligned}
T(X, Y) &= \frac{1}{2}(\nabla^{LC}_X Y - \mathcal{T}(X)(Y) + \Sigma(X)(Y) - \nabla^{LC}_Y X + \mathcal{T}(Y)(X) - \Sigma(Y)(X) - [X, Y]\,) \\
&= \frac{1}{2}(\,-\mathcal{T}(X)(Y) + \Sigma(X)(Y) + \mathcal{T}(Y)(X) - \Sigma(Y)(X)\,) + \frac{1}{2}(\,\nabla^{LC}_X Y - \nabla^{LC}_Y X - [X, Y]\,) \\
&= \frac{1}{2}(\,-\mathcal{T}(X)(Y) + \Sigma(X)(Y) + \mathcal{T}(Y)(X) - \Sigma(Y)(X)\,) + T^{LC}(X, Y) \\
&= \frac{1}{2}(\,-\mathcal{T}(X)(Y) + \Sigma(X)(Y) + \mathcal{T}(Y)(X) - \Sigma(Y)(X)\,)\,,
\end{aligned}$$

where in the last line it is used that the Levi-Civitá connection ∇^{LC} is torsion free. Thus the torsion tensor of affine connection ∇^{T} depends on $\mathcal{T}$ and Σ,

$$T(X,Y,Z) = -g(\mathcal{T}(X)(Y),Z) + g(\mathcal{T}(Y)(X),Z) + g(\Sigma(X)(Y),Z) - g(\Sigma(Y)(X),Z)\,, \tag{2.13}$$

where $T(X,Y,Z) = -T(Y,X,Z)$ apparently. By $T(X,Y,Z) = -T(X,Z,Y)$, T is a 3-form if and only if

$$g(\mathcal{T}(Z)(X),Y) + g(\mathcal{T}(Y)(X),Z) = g(\Sigma(Y)(X),Z) + g(\Sigma(Z)(X),Y) \tag{2.14}$$

holds. Actually, comparing

$$-T(X,Z,Y) = g(\mathcal{T}(X)(Z),Y) - g(\mathcal{T}(Z)(X),Y) - g(\Sigma(X)(Z),Y) + g(\Sigma(Z)(X),Y)$$

to (2.13) give a equation

$$\begin{aligned} & g(\mathcal{T}(Z)(X),Y) + g(\mathcal{T}(Y)(X),Z) - g(\mathcal{T}(X)(Y),Z) - g(\mathcal{T}(X)(Z),Y) \\ = \; & g(\Sigma(Y)(X),Z) + g(\Sigma(Z)(X),Y) - g(\Sigma(X)(Y),Z) - g(\Sigma(X)(Z),Y)\,. \end{aligned}$$

If (2.14) is valid, then we have

$$g(\mathcal{T}(X)(Y),Z) + g(\mathcal{T}(X)(Z),Y) = g(\Sigma(X)(Y),Z) + g(\Sigma(X)(Z),Y)\,.$$

Now we see that

$$\begin{aligned} T(Z,Y,X) &= -g(\mathcal{T}(Z)(Y),X) + g(\mathcal{T}(Y)(Z),X) + g(\Sigma(Z)(Y),X) - g(\Sigma(Y)(Z),X) \\ &= g(\mathcal{T}(Z)(X),Y) - g(\mathcal{T}(X)(Z),Y) - g(\Sigma(Z)(X),Y) + g(\Sigma(X)(Z),Y) \quad (\because X \leftrightarrow Y) \\ &= -g(\mathcal{T}(Y)(X),Z) + g(\Sigma(Y)(X),Z) - g(\mathcal{T}(X)(Z),Y) + g(\Sigma(X)(Z),Y) \quad (\because (2.14)) \\ &= -g(\mathcal{T}(Y)(X),Z) + g(\Sigma(Y)(X),Z) + g(\mathcal{T}(X)(Y),Z) - g(\Sigma(X)(Y),Z) \quad (\because (2)) \\ &= -T(X,Y,Z) \end{aligned}$$

Indeed, when (2.14) is valid, torsion tensor (2.13) is totally skew symmetric tensor. Also, the metric g on M_n is given by a G-invariant form Ω associated with the G-structure $\mathcal{Q}$ and thus the metric has G-invariance.

On associated bundle $\mathcal{Q} \times_G \boldsymbol{R}^n$, Σ and $\mathcal{T}$ belong to $\boldsymbol{R}^n \otimes \mathfrak{g}$ and $\boldsymbol{R}^n \otimes \mathfrak{m}$ respectively:

$$\Sigma \in \boldsymbol{R}^n \otimes \mathfrak{g}\,, \; \mathcal{T} \in \boldsymbol{R}^n \otimes \mathfrak{m}\,.$$

Because $T_pM_n \cong \boldsymbol{R}^n$ $(p \in M_n)$ as vector space, $\Sigma \in T_pM_n \otimes \mathfrak{g}$ and $\mathcal{T} \in T_pM_n \otimes \mathfrak{m}$. The metric g on M_n is given by a G-invariant form Ω associated with the G-structure $\mathcal{Q}$ and thus the metric has G-invariance. Since the metric g is a G-invariant metric, we can define G-invariant maps as follows:

$$\begin{array}{cccc} \Phi \;:\; \boldsymbol{R}^n \otimes \mathfrak{g} & \longrightarrow & \boldsymbol{R}^n \otimes S^2(\boldsymbol{R}^n) \\ \cup & & \cup \\ \Sigma & \longmapsto & \Phi(\Sigma) \end{array} \tag{2.15}$$

$$
\begin{array}{rcl}
\Psi \ : \ \boldsymbol{R}^n \otimes \mathfrak{m} & \longrightarrow & \boldsymbol{R}^n \otimes S^2(\boldsymbol{R}^n) \\
\cup & & \cup \\
\mathcal{T} & \longmapsto & \Psi(\mathcal{T})\,,
\end{array} \tag{2.16}
$$

where $S^2(\boldsymbol{R}^n)$ is a rank 2 symmetric space on $\boldsymbol{R}^n$ and $\Phi(\Sigma)$ and $\Psi(\mathcal{T})$ are defined by

$$
\Phi(\Sigma)(X,Y,Z) := g(\Sigma(Z)(X),Y) + g(\Sigma(Y)(X),Z) \tag{2.17}
$$

$$
\Psi(\mathcal{T})(X,Y,Z) := g(\mathcal{T}(Z)(X),Y) + g(\mathcal{T}(Y)(X),Z)\,. \tag{2.18}
$$

It should be notice that $\Phi(\Sigma)(X,Y,Z)$ and $\Psi(\mathcal{T})(X,Y,Z)$ are symmetric for interchange Y with Z. If we assume the 3-form condition (2.14), then we have

$$
\begin{aligned}
\Psi(\mathcal{T})(X,Y,Z) &= g(\mathcal{T}(Z)(X),Y) + g(\mathcal{T}(Y)(X),Z) \\
&= g(\Sigma(Z)(X),Y) + g(\Sigma(Y)(X),Z) \quad (\because \text{3-form torsion condition (2.14)}) \\
&= \Phi(\Sigma)(X,Y,Z)\,.
\end{aligned}
$$

$\mathcal{Z} = \mathcal{Z}^{LC} - \mathcal{T} + \Sigma$ is an affine connection on a G-structure $\mathcal{Q}$ and the torsion T is given by $\mathcal{T}$ and Σ. The condition that the torsion tensor is a totally skew symmetric is equivalent to $\Psi(\mathcal{T}) = \Phi(\Sigma)$. Since $\dim \mathfrak{g} > \dim \mathfrak{m}$, $K \subseteq L$, where $\Psi(\mathcal{T}) \in K$ and $\Phi(\Sigma) \in L$, if the torsion T is 3-form. It is known that $\mathcal{Q}$ admits $\mathcal{Z}$ with a 3-form torsion T if and only if $K \subseteq L$ [24].

Associated vector bundles $(\mathcal{Q} \times_\rho \boldsymbol{R}^n) \otimes (\mathcal{Q} \times_\rho \mathfrak{m})$ have variety by a chosen representation ρ of the group G. A representation ρ of the group G can be decomposed into irreducible representations of G and then the representation space also can be decomposed into irreducible components. Accordingly the vector bundle $(\mathcal{Q} \times_\rho \boldsymbol{R}^n) \otimes (\mathcal{Q} \times_\rho \mathfrak{m})$ can be splited into irreducible components. Hence on the neighbourhood around a point in M_n the space $\boldsymbol{R}^n \otimes \mathfrak{m}$ is decomposed into irreducible components and $\mathcal{T}$ is decomposed into associated components,

$$
\begin{aligned}
\mathcal{T} \in \boldsymbol{R}^n \otimes \mathfrak{m} &= \bigoplus_{\rho_a} \Lambda_a \\
\mathcal{T} &= \bigoplus_a \mathcal{T}_a\,,
\end{aligned} \tag{2.19}
$$

where ρ_a is a irreducible representation and Λ_a is the associated a-dimensional irreducible components.By decomposing $\boldsymbol{R}^n \otimes \mathfrak{m}$ into irreducible components of the group G, we can characterize the geometric G-structure admitting the G-connection $\mathcal{Z}$ with a 3-form torsion T. The geometric G-structure is also called parallel G-structure and parallel G-structure means $\nabla^T \Omega = 0$ for torsion connection ∇^T. The usage of the parallel G-structure is varied from the standard mean: $\nabla^{LC}\Omega = 0$.

2.1 $SU(3)$-structures

An almost complex manifold (M_{2n}, J) is an $2n$-dimensional real differentiable manifold M_{2n} with an almost complex structure J, which is a tensor field at every point p of M_{2n} such that

$$J : T_pM_{2n} \longrightarrow T_pM_{2n} \ \ (J \text{ is an endomorphism on } T_pM_{2n}) \ , \ J^2 = -1 \, . \tag{2.20}$$

A Nijenhuis tensor N_J is a type (1,2) tensor and it is defined by

$$N_J(X, Y) = [JX, JY] - [X, Y] - J[JX, Y] - J[X, JY] \, , \tag{2.21}$$

where $[\cdot \, , \, \cdot]$ denotes a commutation relation between vector fields. A condition that a Nijenhuis tensor vanishes is known as an integrability condition. If an integrability condition is satisfied, an almost complex manifold is a complex manifold. An almost Hermitian manifold (M_{2n}, g, J) is an almost complex manifold equipped with an almost Hermitian metric g,

$$g(JX, JY) = g(X, Y) \, . \tag{2.22}$$

If the associated Nijenhuis tensor vanishes, the almost Hermitian manifold is said to be a Hermitian manifold. A fundamental 2-form on an almost Hermitian manifold is defined by

$$\kappa(X, Y) = g(X, JY) \tag{2.23}$$

and the 2-form is known as a Kähler form if the Nijenhuis tensor vanishes and κ is a closed form. Then, the Hermitian manifold is said to be Kähler manifold. Since g is invariant by J, κ is also invariant by J,

$$\kappa(JX, JY) = \kappa(X, Y) \, . \tag{2.24}$$

Especially, if the holonomy group of the Levi-Civitá connection ∇ on a Kähler manifold is contained in $SU(n)$, the manifold is known as a Calabi–Yau manifold. According to Ref. [25], if and only if a triplet (κ, J, Υ) satisfies

$$d\kappa = 0 \, , \, d\Upsilon = 0 \, , \tag{2.25}$$

i.e. $\nabla\kappa = 0$, $\nabla J = 0$ and $\nabla\Upsilon = 0$, a manifold $(M_{2n}, \kappa, J, \Upsilon)$ is a Calabi–Yau manifold and thus the condition $d\Upsilon = 0$ implies that $N_J = 0$. A Kähler with torsion (KT) manifold (M_{2n}, g, J) is an almost Hermitian manifold equipped with a 3-form Nijenhuis tensor and the conditions

$$\nabla^T g = 0 \ , \ \nabla^T \kappa = 0 \, , \tag{2.26}$$

where T is a 3-form torsion and it is given by [24]

$$T = -J_* d\kappa + N \, . \tag{2.27}$$

In $N = 0$ case, the 3-form torsion T is said to be the Bismut torsion [23]. If the holonomy of a connection on a KT manifold with vanishing Nijenhuis tensor is contained in $SU(n)$, the manifold is called as a Calabi–Yau with torsion (CYT) manifold. In the case $T = N$, the manifold is also known as the almost Kähler manifold.

A $SU(n)$-structure $\mathcal{Q}_{2n}$ is a principal subbundle of a frame bundle $\mathcal{L}(M_{2n})$ over a $2n$-dimensional almost Hermitian manifold M_{2n}. A $SU(n)$-structure is determined by the choice of an almost complex structure J, a fundamental 2-form κ, and a complex $(n,0)$-form Υ, where κ and Υ satisfy the volume matching condition,

$$\Upsilon \wedge \bar{\Upsilon} = \frac{4}{3} i\, \kappa^n \,. \tag{2.28}$$

The subgroup $SU(n)$ is defined by a stabilizer subgroup of fundamental forms κ and Υ. Then, the structure $\mathcal{Q}_{2n}$ has a structure group $SU(n)$ and the group is a reduced subgroup $SO(2n)$, which is a structure group of $\mathcal{L}(M_{2n})$. From now on, we consider the case $n = 3$. We decompose the Lie algebra $\mathfrak{so}(6)$ into $\mathfrak{su}(3)$ and the orthogonal complement $\mathfrak{m}$, $\mathfrak{so}(6) = \mathfrak{su}(3) \oplus \mathfrak{m}$. A gap of the holonomy group by shrinking from $SO(6)$ to $SU(3)$ is measured by an intrinsic torsion $\mathcal{T}$, which is identified with $\nabla\kappa$, where ∇ denotes Levi-Civitá connection. The torsion $\mathcal{T}$ belongs to $\boldsymbol{R}^6 \otimes \mathfrak{m}$ at local and the space of the intrinsic torsion is decomposed into five components by irreducible representations of $SU(3)$,

$$\boldsymbol{R}^6 \otimes \mathfrak{m} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \oplus \mathcal{W}_5 \,, \tag{2.29}$$

where $\dim_{\boldsymbol{R}} \mathcal{W}_1 = 2\,,\ \dim_{\boldsymbol{R}} \mathcal{W}_2 = 16\,,\ \dim_{\boldsymbol{R}} \mathcal{W}_3 = 12\,,\ \dim_{\boldsymbol{R}} \mathcal{W}_4 = 6\,,\ \dim_{\boldsymbol{R}} \mathcal{W}_5 = 6$ [26](see also [23]). Here $\mathcal{W}_1$ splits into $\mathcal{W}_1^+ \oplus \mathcal{W}_1^- = \boldsymbol{R} \oplus \boldsymbol{R}$ and $\mathcal{W}_2$ decomposes into $\mathcal{W}_2^+ \oplus \mathcal{W}_2^- = \mathfrak{su}(3) \oplus \mathfrak{su}(3)$ [27, 28]. According to Ref. [23], the classification of $SU(3)$-structures is the following:

name	type	condition
Nearly Kähler	$\mathcal{W}_1$	$d\kappa$ is $(3,0)+(0,3)$-form
almost Kähler	$\mathcal{W}_2$	$d\kappa = 0$
Balanced Hermitian	$\mathcal{W}_3$	$N = 0$ and $\delta\kappa = 0$
Conformally Kähler	$\mathcal{W}_4$	$d\kappa = \Theta \wedge \kappa$
KT	$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$\nabla^T g = 0, \nabla^T \kappa = 0$ and T and N are 3-form
Half-flat	$\mathcal{W}_1^- \oplus \mathcal{W}_2^- \oplus \mathcal{W}_3$	$d\mathrm{Im}\Upsilon = 0, d(\kappa \wedge \kappa) = 0$

Here Θ denotes the Lee form and it is defined by

$$\Theta = -J_* \delta\kappa \,, \tag{2.30}$$

where δ is a co-derivative, $\delta = (-1)^{np-p^2+p} * d*$ for p-form on $M_{n=6}$.

We consider a six-dimensional real manifold M_6 admitting a Killing spinor χ satisfying supersymmetric equations in NS-NS sector of type II supergravity theory,

$$\nabla^+_a \chi = 0 \,, \tag{2.31}$$

$$(\,(\partial_a \varphi)\gamma^a - \frac{1}{12} H_{abc} \gamma^{abc}\,)\chi = 0 \,, \tag{2.32}$$

where a 3-form flux H should be closed form. Here $\nabla^+ \chi$ is defined by

$$\nabla^+_a \chi = \nabla_a \chi - \frac{1}{4} \iota_{e_a} H \cdot \chi = \nabla_a \chi - \frac{1}{8} H_{abc} \gamma^{bc} \chi \,. \tag{2.33}$$

When differential forms κ_{ab} and Υ_{abc} are introduced as bilinear form of the Killing spinor,

$$\kappa_{ab} = i\chi^\dagger \gamma_{ab}\chi \ , \ \Upsilon_{abc} = \chi^t \gamma_{abc}\chi \,, \tag{2.34}$$

the supersymmetric equations (3.23) and (3.24) lead to the equations [9]

$$d(e^{-2d\varphi} * \kappa) = 0 \,, \tag{2.35}$$
$$d(e^{-2d\varphi}\Upsilon) = 0 \,, \tag{2.36}$$
$$*H = e^{2d\varphi} d(e^{-2d\varphi}\kappa) \,. \tag{2.37}$$

On the other hand, a six-dimensional CYT manifold permits the $SU(3)$-structure which is defined by

$$d * \kappa - \Theta \wedge *\kappa = 0 \,, \tag{2.38}$$
$$d\Upsilon - \Theta \wedge \Upsilon = 0 \,, \tag{2.39}$$
$$d\kappa - \Theta \wedge \kappa = *T \,, \tag{2.40}$$

where T denotes the Bismut torsion and Θ is the Lee form.Under the identification $H = T$ and $2d\varphi = \Theta$, together with the strong KT condition $dT = 0$, (2.35), (2.36), and (2.37) are clearly equivalent to (2.38), (2.39), and (2.40) Therefore (2.35), (2.36), and (2.37) are defining equations of the $SU(3)$-structure associated with type II supergravity.

A CYT manifold with a closed Bismut torsion and an exact Lee form is a supersymmetric solution of type II supergravity theory [11]. Hence whether the given geometric background has a CYT structure admitting a closed Bismut torsion and an exact Lee form or not is significant on constructing supersymmetric solutions of type II supergravity theory. Especially, the holonomy of a Bismut connection of a KT manifold with a closed Bismut torsion and an exact Lee form should be investigated that it is included in $SU(3)$. According to Ref. [11], if the restricted holonomy is contained in $SU(3)$, then the Ricci form of the Bismut connection vanishes. A Ricci form ρ associated with Bismut connection ∇^+ on a CYT manifold is defined by (see [29])

$$\rho(X,Y) := \frac{1}{2}\sum_{\mu=1}^{6} g(R^+_\#(X,Y)e_\mu, Je_\mu) \quad (\forall X, Y \in T_pM_6) \,, \tag{2.41}$$

where $R^+_\#$ is curvature as type $(1,3)$ tensor field,

$$R^+_\#(X,Y) \ : \ e_\mu \longmapsto R^+_\#(X,Y)e_\mu = R^+_\#(e_\mu, X, Y) \,. \tag{2.42}$$

From now on, we will rewrite the condition $\rho = 0$ as easier condition.

By using definition of a fundamental 2-form κ, we rewrite (2.41) as follows,

$$\rho(X,Y) = \frac{1}{2}\sum_{\mu=1}^{6} \kappa(R^+_\#(X,Y)e_\mu, e_\mu) \,. \tag{2.43}$$

In addition, the right hand side is calculated as follows,

$$\begin{aligned}\sum_{\mu=1}^{6}\kappa(R^{+}_{\#}(X,Y)e_\mu,e_\mu) &= \sum_{\mu=1}^{6}\kappa(X^\nu Y^\rho R^{+}_{\#}(e_\nu,e_\rho)e_\mu,e_\mu) \;=\; X^\nu Y^\rho\sum_{\mu=1}^{6}\kappa(R^{+}_{\#}(e_\mu,e_\nu,e_\rho),e_\mu)\\ &= X^\nu Y^\rho\sum_{\mu=1}^{6}\kappa(R^{+\,\sigma}{}_{\mu\nu\rho}e_\sigma,e_\mu) \;=\; X^\nu Y^\rho\sum_{\mu=1}^{6}R^{+\,\sigma}{}_{\mu\nu\rho}\kappa(e_\sigma,e_\mu)\\ &= \sum_{\mu,\sigma=1}^{6}X^\nu Y^\rho R^{+}{}_{\sigma\mu\nu\rho}\kappa_{\sigma\mu} \;=\; \sum_{\mu,\sigma=1}^{6}X^\nu Y^\rho \mathcal{R}^{+}{}_{\sigma\mu}(e_\nu,e_\rho)\kappa_{\sigma\mu}\\ &= \sum_{\mu,\sigma=1}^{6}\mathcal{R}^{+}{}_{\sigma\mu}\kappa_{\sigma\mu}(X,Y)\,.\end{aligned}$$

Hence we obtain

$$\rho(X,Y)=\frac{1}{2}\sum_{\mu,\sigma=1}^{6}\mathcal{R}^{+}{}_{\sigma\mu}\kappa_{\sigma\mu}(X,Y)\,. \tag{2.44}$$

Here $\mathcal{R}^{T}{}_{\mu\nu}$ is curvature 2-form of a 3-form torsion connection ∇^T and it is defined by

$$\mathcal{R}^{T}{}_{\mu\nu}=d\omega^{T}{}_{\mu\nu}+\sigma_\rho\omega^{T}_{\mu\rho}\wedge\omega^{T}_{\rho\nu}\,, \tag{2.45}$$

where $\omega^{T}_{\mu\nu}$ denotes spin connection with 3-form torsion, which is given by

$$\omega^{T}_{\mu\nu}=\omega_{\mu\nu}+\frac{1}{2}\iota_{e_\mu}\iota_{e_\nu}T\,. \tag{2.46}$$

If $\rho(X,Y)$ vanishes for all X,Y, then $\sum_{\mu,\sigma=1}^{6}R^{+}{}_{\sigma\mu}\kappa_{\sigma\mu}(X,Y)$ vanishes for all X,Y. Thus $\rho=0$ is equivalent to

$$\sum_{\mu,\nu=1}^{6}\mathcal{R}^{+}{}_{\mu\nu}\kappa_{\mu\nu}=0\,. \tag{2.47}$$

This condition is automatically satisfied if the following equation is valid,

$$\frac{1}{2}\sum_{\mu,\nu}\omega^{+}_{\mu\nu}J_{\mu\nu}=0\,. \tag{2.48}$$

A complex structure J is type $(1,1)$ tensor while it can be also written as type $(0,2)$ tensor $J_\#$,

$$g(JY,X)=J_\#(X,Y)\equiv\kappa(X,Y) \tag{2.49}$$

and thus $J_{\mu\nu}=\kappa_{\mu\nu}$. A complex structure J preserves a fundamental 2-form κ,

$$J_*\kappa=\kappa\,. \tag{2.50}$$

The left hand side is calculated as follows in the view of tensor field,

$$J_*\kappa \;=\; J^\rho{}_\mu e_\rho\otimes e^\mu(\kappa_{\sigma\nu}e^\sigma\otimes e^\nu) \;=\; J^\rho{}_\mu\kappa_{\sigma\nu}\delta_\rho{}^\sigma e^\mu\otimes e^\nu \;=\; J^\rho{}_\mu\kappa_{\rho\nu}e^\mu\otimes e^\nu\,.$$

Then, we obtain

$$J^{\rho}{}_{\mu}\kappa_{\rho\nu} = \kappa_{\mu\nu} \ \ , \text{i.e.} \quad J^{\mu}{}_{\rho}J_{\rho\nu} = J_{\mu\nu}\,. \tag{2.51}$$

Now, we can show that (2.48) automatically leads (2.47) as follows:

$$\begin{aligned}
\sum_{\mu,\nu=1}^{6}\mathcal{R}^{+}{}_{\mu\nu}\kappa_{\mu\nu} &= \sum_{\mu,\nu=1}^{6}\mathcal{R}^{+}{}_{\mu\nu}J_{\mu\nu} = \sum_{\mu,\nu=1}^{6}\left(d\omega^{+}_{\mu\nu}+\sum_{\rho}\omega^{+}_{\mu\rho}\wedge\omega^{+}_{\rho\nu}\right)J_{\mu\nu} \\
&= \sum_{\mu,\nu=1}^{6}\left\{d(\omega^{+}_{\mu\nu}J_{\mu\nu})+\sum_{\rho}\omega^{+}_{\mu\rho}J^{\sigma}{}_{\mu}\wedge J_{\sigma\nu}\omega^{+}_{\rho\nu}\right\} \\
&= \sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\mu\rho}J_{\sigma\mu}\wedge\omega^{+}_{\rho\nu}J_{\sigma\nu} = -\sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\mu\rho}J_{\sigma\mu}\wedge\omega^{+}_{\nu\rho}J_{\sigma\nu} \\
&= -\frac{1}{2}\left(\sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\mu\rho}J_{\sigma\mu}\wedge\omega^{+}_{\nu\rho}J_{\sigma\nu}+\sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\mu\rho}J_{\sigma\mu}\wedge\omega^{+}_{\nu\rho}J_{\sigma\nu}\right) \\
&= -\frac{1}{2}\left(\sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\mu\rho}J_{\sigma\mu}\wedge\omega^{+}_{\nu\rho}J_{\sigma\nu}-\sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\nu\rho}J_{\sigma\nu}\wedge\omega^{+}_{\mu\rho}J_{\sigma\mu}\right) \\
&= -\frac{1}{2}\left(\sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\mu\rho}J_{\sigma\mu}\wedge\omega^{+}_{\nu\rho}J_{\sigma\nu}-\sum_{\mu,\nu,\rho,\sigma=1}^{6}\omega^{+}_{\mu\rho}J_{\sigma\mu}\wedge\omega^{+}_{\nu\rho}J_{\sigma\nu}\right) \\
&= 0\,.
\end{aligned}$$

Thus (2.47) is trivially assured when (2.48) is valid.

2.2 HKT-structure

We consider an almost Hermitian manifold $(M_{4n}, g, J_1, J_2, J_3)$ which has three almost complex structure J_1, J_2, and J_3 where the almost complex structures satisfy the conditions

$$J_1J_2 = J_3\ ,\ J_3J_1 = J_2\ ,\ J_2J_3 = J_1\ ,\ {J_a}^2 = -1 \quad (a = 1,2,3) \tag{2.52}$$

and Hermitian conditions

$$g(J_aX\ ,\ J_aY) = g(X,Y)\,. \tag{2.53}$$

Then, the fundamental 2-forms κ_a $(a = 1,2,3)$ are defined by

$$\kappa_a(X,Y) = g(X,J_aY)\,. \tag{2.54}$$

A hyper Kähler manifold is the manifold $(M_{4n}, g, J_1, J_2, J_3)$ admitting the conditions that Nijenhuis tensors associated with three almost complex structures vanish

$$N_{J_1} = 0\ ,\ N_{J_2} = 0\ ,\ N_{J_3} = 0 \tag{2.55}$$

and the complex structures are parallel for a covariant derivative of Levi-Civitá connection,

$$\nabla J_1 = 0\ ,\ \nabla J_2 = 0\ ,\ \nabla J_3 = 0\,. \tag{2.56}$$

The examples of 4-dimensional hyper Kähler metrics are known as Euclidean Taub-NUT metric, Eguchi–Hanson metric, Gibbons–Hawking metric and Atiyah–Hitchin metric.

A hyper Kähler with torsion (HKT) manifold is the manifold $(M_{4n}, g, J_1, J_2, J_3)$ obeying the conditions that Nijenhuis tensors N_{J_a} $(a = 1, 2, 3)$ vanish and there exist a unique connection with 3-form torsion T such that

$$\nabla^T J_1 = 0\ ,\ \nabla^T J_2 = 0\ ,\ \nabla^T J_3 = 0\,, \tag{2.57}$$

where the torsion is given by

$$T = J_1 d\kappa_1 = J_2 d\kappa_2 = J_3 d\kappa_3\,. \tag{2.58}$$

4-dimensional HKT metrics are constructed by conformal transformation of hyper Kähler metrics, $g_{HKT} = \Phi g_{HK}$.

2.3 G_2-structures

We consider $\boldsymbol{R}^7$ equipped with an orientation and inner product. The space of 2-forms on $\boldsymbol{R}^4$ $\wedge^2\boldsymbol{R}^4$ can be separated by the eigenvalue as follows,

$$\wedge^2\boldsymbol{R}^4 = \wedge^2_+\boldsymbol{R}^4 \oplus \wedge^2_-\boldsymbol{R}^4\,. \tag{2.59}$$

Then, the space $\boldsymbol{R}^7$ is isomorphic to the space $\wedge^2_\pm\boldsymbol{R}^4$ as vector space. Namely, $\wedge^2_\pm\boldsymbol{R}^4$ is the total space over the base space $\boldsymbol{R}^4$ with fiber $\Gamma(\wedge^2_\pm\boldsymbol{R}^4)$, where $\Gamma(\wedge^2_\pm\boldsymbol{R}^4)$ is a space of 2-forms,

$$\Gamma(\wedge^2_\pm\boldsymbol{R}^4) = \{\alpha \in \wedge^2\boldsymbol{R}^4\ :\ *\alpha = -\alpha\}\,. \tag{2.60}$$

Let us introduce a coordinate (y_1, y_2, y_3, y_4) on $\boldsymbol{R}^4$ and 2-forms under the volume form dy^{1234}

$$\beta_1 = dy^{12} - dy^{34}\ ,\ \beta_2 = dy^{42} - dy^{13}\ ,\ \beta_3 = dy^{14} - dy^{23}\,, \tag{2.61}$$

where dy^{ij} means $dy^i \wedge dy^j$ for short. There exist a projection π_1 between $\wedge^2\boldsymbol{R}^4$ and $\boldsymbol{R}^4$,

$$\pi_1\ :\ \wedge^2_-\boldsymbol{R}^4 \longrightarrow \boldsymbol{R}^4\,.$$

The fiber of $\wedge^2\boldsymbol{R}^4$ is given by

$$\pi_1^{-1}(p) = \{\alpha_p \in \wedge^2_-\boldsymbol{R}^4\ :\ *\alpha_p = -\alpha_p\} \tag{2.62}$$

and we define a fiber coordinate (a_1, a_2, a_3) on the total space $\wedge^2_-\boldsymbol{R}^4$ as follows,

$$\beta = a_1\beta_1 + a_2\beta_2 + a_3\beta_3\,. \tag{2.63}$$

Now, since $\boldsymbol{R}^7$ is isomorphic to $\wedge^2_- \boldsymbol{R}^4$, we can define a 3-form Ω_0 on $\boldsymbol{R}^7$ by

$$\Omega_0 = da^{123} - \sum_{i=1}^{3} da^i \wedge \beta^i \,. \tag{2.64}$$

The written down form is given by

$$\Omega_0 = da^{123} - da^1 \wedge dy^{12} + da^1 \wedge dy^{34} - da^2 \wedge dy^{42} + da^2 \wedge dy^{13} - da^3 \wedge dy^{14} + da^3 \wedge dy^{23} \,. \tag{2.65}$$

When we define an another coordinate $(x_1, \cdots, x_7)$ as

$$x_1 = a_1 \,,\; x_2 = a_2 \,,\; x_3 = a_3 \,,\; x_4 = y_1 \,,\; x_5 = y_4 \,,\; x_6 = y_3 \,,\; x_7 = -y_2 \,, \tag{2.66}$$

we obtain

$$\Omega_0 = dx^{123} + dx^{147} + dx^{165} + dx^{257} + dx^{246} + dx^{354} + dx^{367} \,. \tag{2.67}$$

The 3-form Ω_0 define the exceptional Lie group G_2 as stabilizer subgroup of $GL(7; \boldsymbol{R})$,

$$G_2 = \{ s \in GL(7; \boldsymbol{R}) \;:\; \rho^*(s)\Omega_0 = \Omega_0 \} \,, \tag{2.68}$$

where the 3-form Ω_0 is said to be the fundamental 3-form.

We can lift up this to a 7-dimensional Riemannian manifold M_7. Namely, the group G_2 is the isotropy group of the fundamental 3-form Ω,

$$G_2 = \{ s \in SO(7) \;:\; \rho^*(s)\Omega = \Omega \} \,, \tag{2.69}$$

where by using orthonormal basis $\{e^1, \cdots, e^7\}$ on T_pM_7, the 3-form Ω is expressed locally as follows:

$$\Omega = e^{123} + e^{147} + e^{165} + e^{246} + e^{257} + e^{354} + e^{367} \,. \tag{2.70}$$

Given the volume form of M_7 by $\mathrm{Vol.}(M_7) = e^{1234567}$, the Hodge dual of Ω is

$$*\Omega = e^{4567} + e^{2356} - e^{2347} + e^{1357} + e^{1346} - e^{1267} + e^{1245} \,. \tag{2.71}$$

In the view of a vector space, we identify the Lie algebra $\mathfrak{so}(7)$ with the space of all 2-form,

$$\mathfrak{so}(7) \cong \wedge^2(\boldsymbol{R}^7) = \left\{ A = \sum_{\mu<\nu} \beta_{\mu\nu} e^{\mu\nu} \right\} \,. \tag{2.72}$$

The Lie algebra $\mathfrak{g}_2$ of the Lie group G_2 is given by

$$\mathfrak{g}_2 = \{ a \in \mathfrak{so}(7) \;:\; \varrho(a)\Omega = 0 \} = \left\{ A = \sum_{\mu<\nu} \beta_{\mu\nu} e^{\mu\nu} \in \wedge^2(\boldsymbol{R}^7) \;:\; A \wedge *\Omega = 0 \right\} \,, \tag{2.73}$$

where the map ϱ is a faithful representation, $\varrho : \mathfrak{so}(7) \longrightarrow \mathfrak{so}(V)$,

$$\begin{aligned}
\varrho(a)\Omega &= A \wedge *\Omega = \sum_{\mu<\nu} \beta_{\mu\nu} e^{\mu\nu} \wedge (e^{4567} + e^{2356} - e^{2347} + e^{1357} + e^{1346} - e^{1267} + e^{1245}) \\
&= (\beta_{23} + \beta_{47} - \beta_{56}) e^{234567} + (\beta_{13} - \beta_{46} - \beta_{57}) e^{134567} + (\beta_{12} - \beta_{45} + \beta_{67}) e^{124567} \\
&\quad + (\beta_{17} + \beta_{26} - \beta_{35}) e^{123567} + (\beta_{16} - \beta_{27} - \beta_{34}) e^{123467} + (\beta_{15} - \beta_{24} + \beta_{37}) e^{123457} \\
&\quad + (\beta_{14} + \beta_{25} + \beta_{36}) e^{123456} .
\end{aligned}$$

Therefore we can restate the definition of the Lie algebra $\mathfrak{g}_2$

$$\begin{aligned}
&\beta_{23} + \beta_{47} - \beta_{56} = 0 \ , \ \beta_{13} - \beta_{46} - \beta_{57} = 0 \ , \ \beta_{12} - \beta_{45} + \beta_{67} = 0 \ , \ \beta_{17} + \beta_{26} - \beta_{35} = 0 \, , \\
&\beta_{16} - \beta_{27} - \beta_{34} = 0 \ , \ \beta_{15} - \beta_{24} + \beta_{37} = 0 \ , \ \beta_{14} + \beta_{25} + \beta_{36} = 0 \, . \qquad (2.74)
\end{aligned}$$

Here the condition $A \wedge *\Omega = 0$ is equivalent to $*(A \wedge \Omega) = A$. Actually, $*(A \wedge \Omega)$ is given by

$$\begin{aligned}
*(A \wedge \Omega) &= (\beta_{45} - \beta_{12}) e^{67} - (\beta_{46} - \beta_{13}) e^{57} + (\beta_{47} + \beta_{23}) e^{56} + (\beta_{56} - \beta_{23}) e^{47} - (\beta_{57} - \beta_{13}) e^{46} \\
&\quad + (\beta_{67} + \beta_{12}) e^{45} - (-\beta_{24} + \beta_{15}) e^{37} + (-\beta_{25} - \beta_{14}) e^{36} - (-\beta_{26} - \beta_{17}) e^{35} \\
&\quad + (-\beta_{27} + \beta_{16}) e^{34} + (-\beta_{34} + \beta_{16}) e^{27} - (-\beta_{35} + \beta_{17}) e^{26} + (-\beta_{36} - \beta_{14}) e^{25} \\
&\quad - (-\beta_{37} - \beta_{15}) e^{24} + (\beta_{56} - \beta_{47}) e^{23} - (-\beta_{35} + \beta_{26}) e^{17} + (\beta_{34} + \beta_{27}) e^{16} \\
&\quad - (\beta_{37} - \beta_{24}) e^{15} + (-\beta_{36} - \beta_{25}) e^{14} - (-\beta_{57} - \beta_{46}) e^{13} + (-\beta_{67} + \beta_{45}) e^{12} .
\end{aligned}$$

Hence we see that the condition $*(A \wedge \Omega) = A$ lead us to (2.74). Thus (2.73) is rewritten as follows,

$$\begin{aligned}
\mathfrak{g}_2 &= \left\{ A = \sum_{\mu<\nu} \beta_{\mu\nu} e^{\mu\nu} \in \wedge^2(\boldsymbol{R}^7) \ : \ A \wedge *\Omega = 0 \right\} \\
&= \left\{ A = \sum_{\mu<\nu} \beta_{\mu\nu} e^{\mu\nu} \in \wedge^2(\boldsymbol{R}^7) \ : \ *(A \wedge \Omega) = A \right\} . \qquad (2.75)
\end{aligned}$$

A G_2-structure $\mathcal{Q}_7$ is a principal subbundle of a frame bundle $\mathcal{L}(M_7)$ over a seven-dimensional oriented Riemannian manifold M_7. A G_2-structure is determined by the choice of a fundamental 3-form Ω. As was already mentioned, the subgroup G_2 is defined by a stabilizer subgroup of Ω. Then, the structure $\mathcal{Q}_7$ has a structure group G_2 and the group is a reduced subgroup of $SO(7)$, which is a structure group $\mathcal{L}(M_7)$. A seven dimensional Riemannian manifold admitting G_2-structure is called a G_2-manifold. If the fundamental 3-form Ω is parallel with respect to Levi-Civitá connection, $\nabla\Omega = 0$, then the holonomy is contained in G_2 and also the G_2-manifold called parallel G_2-manifold. In addition, the condition $\nabla\Omega = 0$ is equivalent to $d\Omega = d * \Omega = 0$.

We decompose the Lie algebra $\mathfrak{so}(7)$ into $\mathfrak{g}_2$ and the orthogonal $\mathfrak{m}$, $\mathfrak{so}(7) = \mathfrak{g}_2 \oplus \mathfrak{m}$. A failure of the holonomy group by shrinking from $SO(7)$ to G_2 is measured by an intrinsic torsion $\mathcal{T}$, which is identified with $\nabla\Omega$, where ∇ denotes Levi-Civitá connection. The torsion $\mathcal{T}$ belongs to $\boldsymbol{R}^7 \otimes \mathfrak{m}$

locally and the space of the intrinsic torsion is decomposed into four irreducible components by irreducible representations of G_2,

$$\boldsymbol{R}^7 \otimes \mathfrak{m} = \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \,, \tag{2.76}$$

where $\dim \mathcal{W}_1 = 1$, $\dim \mathcal{W}_2 = 14$, $\dim \mathcal{W}_3 = 27$, $\dim \mathcal{W}_4 = 7$ [13](see also Ref. [30]). Since $\mathcal{T}$ belongs to $\boldsymbol{R}^7 \otimes \mathfrak{m}$, we have

$$\nabla \Omega = \varrho(\mathcal{T})(\Omega) \,\in\, \mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4 \,. \tag{2.77}$$

On a G_2-manifold, the condition that the $\mathcal{W}_2$ component of the intrinsic torsion $\mathcal{T}$ vanishes is equivalent to the condition that there exist an unique affine connection ∇^T with 3-form torsion preserving the G_2-structure, $\nabla^T \Omega = 0$ [24]. We consider the G_2-structure of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, which is characterized by the condition $\mathcal{T} \in \mathcal{W}_1 \oplus \mathcal{W}_4 \oplus \mathcal{W}_3$. Then, we decompose $\mathcal{T}$ into three parts,

$$\mathcal{T} = \lambda \, \mathrm{Id} \oplus \Theta \oplus \mathcal{T}^3_{27} \,, \tag{2.78}$$

where λ is a function on M_7. The $\mathcal{W}_1$ component $\lambda \, \mathrm{Id}$ is the map $\lambda \, \mathrm{Id} \,:\, \boldsymbol{R}^7 \to \mathfrak{m}$, which is defined by

$$\lambda \, \mathrm{Id}(X) := \frac{\lambda}{12} \iota_X \Omega \tag{2.79}$$

where $X \in \boldsymbol{R}^7$. Let us introduce the orthonormal projection $\mathfrak{pr}_m \,:\, \mathfrak{so}(7) = \wedge^2(\boldsymbol{R}^7) \to \mathfrak{m}$ by the formula

$$\mathfrak{pr}_m(\alpha_2) = \frac{1}{3} \sum_{i=1}^{7} (\alpha_2, \iota_{e_i}\Omega) \cdot \iota_{e_i}\Omega \,, \tag{2.80}$$

where $\left(\frac{1}{\sqrt{3}}\iota_{e_1}\Omega, \cdots, \frac{1}{\sqrt{3}}\iota_{e_7}\Omega\right)$ is an orthonormal basis of $\mathfrak{m} \subset \mathfrak{so}(7)$. The $\mathcal{W}_4$ component Θ is the 1-form $\Theta \,:\, \boldsymbol{R}^7 \to \mathfrak{m}$, which is defined by

$$\Theta(X) := \frac{1}{4} \mathfrak{pr}_m(\Theta \wedge X^*) = \frac{1}{12} (\Theta \wedge X^* \,,\, \iota_{e_i}\Omega) \cdot \iota_{e_i}\Omega \,. \tag{2.81}$$

The $\mathcal{W}_3$ component $\mathcal{T}^3_{27}$ is the 3-form $\mathcal{T}^3_{27} \,:\, \boldsymbol{R}^7 \to \mathfrak{m}$, which is defined by

$$\mathcal{T}^3_{27}(X) := \frac{1}{2} \mathfrak{pr}_m(\iota_X \mathcal{T}^3_{27}) = \frac{1}{6} \sum_{i=1}^{7} (\iota_X \mathcal{T}^3_{27}, \iota_{e_i}\Omega) \cdot \iota_{e_i}\Omega \,. \tag{2.82}$$

The equation (2.80) is rewritten as follows,

$$\mathfrak{pr}_m(\alpha_2) = \iota_Y \Omega \;,\; Y = \sum_{i=1}^{7} Y_i e_i \equiv \frac{1}{3} \sum_{i=1}^{7} (\alpha_2, \iota_{e_i}\Omega) e_i \tag{2.83}$$

and thus we have $\alpha_2 = \iota_Y \Omega$. According to Ref. [31], $\varrho(\alpha_2)$ is given by

$$\varrho(\alpha_2)(\Omega) = -3 \iota_Y * \Omega \,. \tag{2.84}$$

Thus (2.77) without a $\mathcal{W}_2$ component is given by

$$\nabla_X \Omega = -\frac{\lambda}{4}(\iota_X * \Omega) - \sum_{i=1}^{7} \left(\frac{1}{4}\Theta \wedge X^* + \frac{1}{2}\iota_X \mathcal{T}^3_{27} \, , \, \iota_{e_i}\Omega \right) \cdot (\iota_{e_i} * \Omega) \, . \tag{2.85}$$

Spaces of differential forms are splited into irreducible components of G_2 representations and the spaces are given by [24, 30]

$$\begin{aligned} &\wedge^2_7 = \{\alpha^2 \in \wedge^2(\boldsymbol{R}^7) \ : \ *(\Omega \wedge \alpha^2) = -2\alpha^2\} = \{\iota_X \Omega \ : \ X \in \boldsymbol{R}^7\} = \{*(*\Omega \wedge \alpha^1) \ : \ \alpha^1 \in \wedge^1_7\} \, , \\ &\wedge^2_{14} = \{\alpha^2 \in \wedge^2(\boldsymbol{R}^7) \ : \ *(\Omega \wedge \alpha^2) = \alpha^2\} \, , \end{aligned} \tag{2.86}$$

$$\begin{aligned} &\wedge^3_1 = \{t\,\Omega \ : \ t \in \boldsymbol{R}\} \, , \\ &\wedge^3_7 = \{*(\Omega \wedge \alpha^1) \ : \ \alpha^1 \in \wedge^1(\boldsymbol{R}^7)\} = \{\iota_X * \Omega \ : \ X \in \boldsymbol{R}^7\} \, , \\ &\wedge^3_{27} = \{\alpha^3 \in \wedge^3(\boldsymbol{R}^7) \ : \ \alpha^3 \wedge \Omega = 0 \ , \ \alpha^3 \wedge *\Omega = 0\} \, , \end{aligned} \tag{2.87}$$

$$\begin{aligned} &\wedge^4_1 = \{t * \Omega \ : \ t \in \boldsymbol{R}\} \, , \\ &\wedge^4_7 = \{\alpha_1 \wedge \Omega \mid \alpha_1 \in \wedge^1(\boldsymbol{R}^7)\} \, , \\ &\wedge^4_{27} = \{\alpha^4 \in \wedge^3(\boldsymbol{R}^7) \ : \ \alpha^4 \wedge \Omega = 0\} \, . \end{aligned} \tag{2.88}$$

Then, on the components $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{W}_3$, and $\mathcal{W}_4$ they are isomorphic to $\wedge^0_1$, $\wedge^2_{14}$, $\wedge^3_{27}$, and $\wedge^1_7$ as vector spaces respectively, $\mathcal{W}_1 \cong \wedge^0_1$, $\mathcal{W}_2 \cong \wedge^2_{14}$, $\mathcal{W}_3 \cong \wedge^3_{27}$, and $\mathcal{W}_4 \cong \wedge^1_7$. We consider coderivative of Ω and it is given by

$$\delta\Omega := -\sum_{i=1}^{7}(\iota_{e_j}\nabla_{e_j}\Omega) = \sum_{i,j=1}^{7} \left(\frac{1}{4}\Theta \wedge e_j + \frac{1}{2}(\iota_{e_j}\mathcal{T}^3_{27}) \, , \, \iota_{e_i}\Omega \right) \cdot \iota_{e_j}\iota_{e_i} * \Omega \, , \tag{2.89}$$

where in the second equal sigh, we used the equation

$$\sum_{j=1}^{7} \iota_{e_j}\iota_{e_j} * \Omega = 0 \, . \tag{2.90}$$

The linear map form the vector space $\wedge^3_{27}$ to vector space $\wedge^2 = \wedge^2_7 \oplus \wedge^2_{14}$, i.e.

$$\mathcal{T}^3_{27} \in \wedge^3_{27} \longmapsto \sum_{i=1}^{7}(\iota_{e_j}\mathcal{T}^3_{27} \, , \, \iota_{e_i}\Omega) \cdot \iota_{e_j}\iota_{e_i} * \Omega \in \wedge^2_7 \oplus \wedge^2_{14} \, ,$$

is the zero-map on the vector spaces and thus we have

$$\sum_{i=1}^{7}(\iota_{e_j}\mathcal{T}^3_{27} \, , \, \iota_{e_i}\Omega) \cdot \iota_{e_j}\iota_{e_i} * \Omega = 0 \, . \tag{2.91}$$

Also, the linear map

$$\Theta \in \wedge^1_7 = \boldsymbol{R}^7 \longmapsto \sum_{i,j=1}^{7} (\Theta \wedge e^j \, , \, \iota_{e_i}\Omega) \cdot \iota_{e_j}\iota_{e_i} * \Omega \in \wedge^2 = \wedge^2_7 \oplus \wedge^2_{14}$$

is the identity map between the vector spaces and thus the image of the map contained in $\wedge^2_7$,

$$\Theta \longmapsto \iota_{\Theta_*}\Omega \, .$$

Hence we obtain

$$\sum_{i,j=1}^{7} (\Theta \wedge e^j \, , \, \iota_{e_i}\Omega) \cdot \iota_{e_j}\iota_{e_i} * \Omega = -4\iota_{\Theta_*}\Omega \, . \tag{2.92}$$

Consequently, $\delta\Omega$ is given by

$$\delta\Omega = -\iota_\Theta\Omega = - * (\Theta \wedge *\Omega) \, . \tag{2.93}$$

We consider exterior derivative of Ω and it is given by

$$d\Omega := \sum_{i=1}^{7} e^i \wedge \nabla_{e_i}\Omega = -\frac{\lambda}{4}\sum_{i=1}^{7} e^i \wedge (\iota_{e_i} * \Omega) - \sum_{i=1}^{7}\left(\frac{1}{4}\Theta \wedge e^i + \frac{1}{2}\iota_{e_i}\mathcal{T}^3_{27} \, , \, \iota_{e_j}\Omega\right) \cdot (e^i \wedge \iota_{e_j} * \Omega) \, . \tag{2.94}$$

On the 1-st term of $d\Omega$, it is in proportion to $*\Omega$ and the constant of the proportionality is equal to 4 because there are four each $e^i \ \ (i = 1, \cdots, 7)$ in $*\Omega$. Thus we obtain

$$-\frac{\lambda}{4}\sum_{i=1}^{7} e^i \wedge (\iota_{e_i} * \Omega) = -\lambda * \Omega \, . \tag{2.95}$$

The linear map from $\wedge^3_{27}$ to $\wedge^4 = \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}$, which is defined by

$$\mathcal{T}^3_{27} \in \wedge^3_{27} \longmapsto \sum_{i=1}^{7}\left(\frac{1}{2}\iota_{e_i}\mathcal{T}^3_{27} \, , \, \iota_{e_j}\Omega\right) \cdot (e^i \wedge \iota_{e_j} * \Omega) \in \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27} \, ,$$

is the identity map between the vector spaces $\wedge^3_{27}$ and $\wedge^4_{27}$, where $*\mathcal{T}^3_{27} \in \wedge^4_{27}$. Thus we have

$$\sum_{i=1}^{7}\left(\frac{1}{2}\iota_{e_i}\mathcal{T}^3_{27} \, , \, \iota_{e_j}\Omega\right) \cdot (e^i \wedge \iota_{e_j} * \Omega) = -2 * \mathcal{T}^3_{27} \, . \tag{2.96}$$

The linear map from $\wedge^1_7$ to $\wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27}$, i.e.,

$$\Theta \in \wedge^1_7 = \boldsymbol{R}^7 \longmapsto \sum_{i=1}^{7}\left(\frac{1}{4}\Theta \wedge e^i \, , \, \iota_{e_j}\Omega\right) \cdot (e^i \wedge \iota_{e_j} * \Omega) \in \wedge^4_1 \oplus \wedge^4_7 \oplus \wedge^4_{27} \, ,$$

is the identity map from $\wedge^1_7 = \wedge^3_7$ to $\wedge^4_7$ and the image of Θ, i.e., $\Theta \wedge \Omega$, is in $\wedge^4_7$. Therefore we obtain

$$\sum_{i=1}^{7}\left(\frac{1}{4}\Theta \wedge e^i \, , \, \iota_{e_j}\Omega\right) \cdot (e^i \wedge \iota_{e_j} * \Omega) = -3\Theta \wedge \Omega \, . \tag{2.97}$$

Consequently, $d\Omega$ is given by

$$d\Omega = -\lambda * \Omega + *\mathcal{T}^3_{27} + \frac{3}{4}(\Theta \wedge \Omega)\,. \tag{2.98}$$

We rewrite 3-form torsion T, (2.13) with the condition (2.14) by fundamental 3-form Ω. Let us introduce the map $\varsigma\,:\,\wedge^1_7 \oplus \wedge^3_{27} \to \boldsymbol{R}^7 \otimes \mathfrak{g}_2$ by the formula

$$\varsigma(\mathcal{T})(X) = \frac{1}{2}\mathfrak{pr}_{g_2}\left(\iota_X \mathcal{T}^3_{27} - \frac{1}{4}\Theta \wedge X^*\right)\,, \tag{2.99}$$

where $\mathfrak{pr}_{g_2}(\alpha_2) = \alpha_2 - \mathfrak{pr}_m(\alpha_2)$. Also, we define the 1-form $\mathcal{T}^*$ as follows,

$$\mathcal{T}^*(X) = \mathcal{T}(X) - \varsigma(\mathcal{T})(X) \tag{2.100}$$

and it takes the following explicit form,

$$\mathcal{T}^*(X) = \frac{\lambda}{12}\iota_X\Omega + \frac{1}{2}\iota_X\mathcal{T}^3_{27} + \frac{3}{8}\mathfrak{pr}_m(\Theta \wedge X^*) - \frac{1}{8}\Theta \wedge X^*\,. \tag{2.101}$$

Then, the 3-form torsion T is given by

$$T(X, Y, Z) = -g(\mathcal{T}^*(X)(Y)\,,\,Z) + g(\mathcal{T}^*(Y)(X)\,,\,Z) \tag{2.102}$$

and the direct calculation lead

$$T = -\frac{\lambda}{6}\Omega - \mathcal{T}^3_{27} - \frac{1}{4}\iota_{\Theta_*} * \Omega\,. \tag{2.103}$$

When we do inner product with $*\Omega$ in (2.98), the each terms in the right hand side are given by

$$\begin{aligned}
(*\Omega\,,\,*\Omega) &= \int_{M_7} *\Omega \wedge \Omega = \int_{M_7} 7\mathrm{Vol.}(M_7)\cdot 1 = 7\,,\\
(*\mathcal{T}^3_{27}\,,\,*\Omega) &= \int_{M_7} *\mathcal{T}^3_{27} \wedge \Omega = 0 \quad (\because *\mathcal{T}^3_{27} \in \wedge^4_{27})\,,\\
(\Theta \wedge \Omega\,,\,*\Omega) &= \int_{M_7} \Theta \wedge \Omega \wedge \Omega = 0\,.
\end{aligned}$$

Then, we obtain

$$\lambda = -\frac{1}{7}(d\Omega\,,\,*\Omega) = -\frac{1}{7} * (d\Omega \wedge \Omega)\,. \tag{2.104}$$

Acting the Hodge dual operator in (2.98), we have

$$\mathcal{T}^3_{27} = *d\Omega + \lambda\Omega - \frac{3}{4} * (\Theta \wedge \Omega)\,. \tag{2.105}$$

From the component $\wedge^2_7$, we obtain the relation

$$\iota_{\Theta_*}\Omega = *(\Theta \wedge *\Omega)\,. \tag{2.106}$$

Consequently, on the G_2-manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, there exist unique affine connection with 3-form torsion, which is given by

$$T = \frac{1}{6} * (d\Omega \wedge \Omega)\, \Omega - *d\Omega + *(\Theta \wedge \Omega) \tag{2.107}$$

and the associate G_2-structure classified by

$$\delta\Omega = - * (\Theta \wedge *\Omega)\ ,\ d\Omega = \frac{1}{7} * (d\Omega \wedge \Omega) * \Omega + *\mathcal{T}^3_{27} + \frac{3}{4}(\Theta \wedge \Omega)\,. \tag{2.108}$$

According to Refs. [30, 32], the classification of G_2-structures is the following:

Name	type	conditions
Parallel G_2	$\{0\}$	$d\Omega = 0\,,\ d*\Omega = 0$
Nearly parallel G_2	$\mathcal{W}_1$	$d\Omega = -\lambda * \Omega$ and $d * \Omega = 0$
Almost parallel G_2	$\mathcal{W}_2$	$d\Omega = 0$
Balanced G_2	$\mathcal{W}_3$	$\delta\Omega = 0$ and $\lambda = 0$
locally conformally parallel G_2	$\mathcal{W}_4$	$d\Omega = -\lambda * \Omega + \frac{3}{4}\Theta \wedge \Omega$ and $d * \Omega = \Theta \wedge *\Omega$
	$\mathcal{W}_1 \oplus \mathcal{W}_2$	$d\Omega = -\lambda * \Omega$
cocalibrated G_2	$\mathcal{W}_1 \oplus \mathcal{W}_3$	$\delta\Omega = 0$
Locally conformally nearly G_2	$\mathcal{W}_1 \oplus \mathcal{W}_4$	$d\Omega = -\lambda * \Omega$ and $d * \Omega = \Theta \wedge *\Omega$
	$\mathcal{W}_2 \oplus \mathcal{W}_3$	$\lambda = 0$ and $\Theta = 0$
Locally conformally parallel G_2	$\mathcal{W}_2 \oplus \mathcal{W}_4$	$d\Omega = \frac{3}{4}\Theta \wedge \Omega$
	$\mathcal{W}_3 \oplus \mathcal{W}_4$	$\lambda = 0$ and $d * \Omega = \Theta \wedge \Omega$
	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3$	$\Theta = 0$
	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_4$	$d\Omega = -\lambda * \Omega + \frac{3}{4}\Theta \wedge \Omega$
G_2T	$\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$d * \Omega = \Theta \wedge *\Omega$
	$\mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	$\lambda = 0$
	$\mathcal{W}_1 \oplus \mathcal{W}_2 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$	no relations

Θ denotes the Lee form, which is defined by

$$\Theta = -\frac{1}{3} * (*d\Omega \wedge \Omega)\,. \tag{2.109}$$

Let us rewrite the Lee form. The integration of $*d\Omega \wedge \Omega$ on M_7 is given by

$$\begin{aligned}
\int_{M_7} *d\Omega \wedge \Omega \;&=\; \int_{M_7} *d(**\Omega) \wedge **\Omega \;=\; \int_{M_7} *d*(*\Omega) \wedge *(*\Omega) \\
&=\; \int_{M_7} \delta * \Omega \wedge *(*\Omega) \\
&=\; \int_{M_7} *\Omega \wedge \delta * (*\Omega) \;=\; -\int_{M_7} *\Omega \wedge \delta\Omega \\
&=\; \int_{M_7} \delta\Omega \wedge *\Omega\,,
\end{aligned}$$

where from the line 2 to the line 3 we use

$$\int_{M_7} \delta(*\Omega \wedge \Omega) = 7 \int_{M_7} \delta \mathrm{Vol.}(M_7) = \int_{M_7} *d\,1 = 0\,. \tag{2.110}$$

Thus we obtain

$$\Theta = \frac{1}{3} * (\delta\Omega \wedge *\Omega)\,. \tag{2.111}$$

If the Lee form Θ vanishes, i.e. $\delta\Omega = 0$, and $\mathcal{T}^3_{27} = 0$ in the G_2-structure of the class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, then the G_2-structure reduces to the subclass $\mathcal{W}_1$. The G_2-manifold of type $\mathcal{W}_1$ is known as a nearly parallel G_2-manifold. If the Lee form Θ vanishes and $\lambda = 0$ in class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, the G_2-structure reduce to the subclass $\mathcal{W}_3$, which is said to be a balanced G_2-structure. If the component $\mathcal{T}^3_{27}$ is zero in the G_2-manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, the G_2-manifold has the G_2-structure of type $\mathcal{W}_4$. The G_2-manifold of type $\mathcal{W}_4$ is called as locally conformally parallel G_2-manifold. The condition $\Theta = 0$ in the G_2-manifold of class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ gives the subclass $\mathcal{W}_1 \oplus \mathcal{W}_3$ and the G_2-manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_3$ is known as cocalibrated G_2-manifold. When the component $\mathcal{T}^3_{27}$ vanishes and $\Theta \wedge \Omega = 0$ in type $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$, we obtain the subclass $\mathcal{W}_1 \oplus \mathcal{W}_4$. The G_2-manifold of type $\mathcal{W}_1 \oplus \mathcal{W}_4$ is called as locally conformally nearly G_2-manifold. We will investigate the class $\mathcal{W}_3 \oplus \mathcal{W}_4$ because the class is associated with type II or heterotic supergravity theory.

We consider a seven dimensional real oriented manifold admitting the type $\mathcal{W}_3 \oplus \mathcal{W}_4$ G_2-structure. The G_2-structure is defined by the following equations [9, 24]:

$$d * \Omega - \Theta \wedge *\Omega = 0\,, \tag{2.112}$$
$$d\Omega \wedge \Omega = 0\,. \tag{2.113}$$

Also, there exist an unique connection ∇^T such that $\nabla^T \Omega = 0$ [24] and the torsion is given by

$$d\Omega - \Theta \wedge \Omega = - * T\,. \tag{2.114}$$

By the way, the G_2-structure associated with type II supergravity theory is defined by

$$e^{2\varphi} d(e^{-2\varphi} * \Omega) = 0\,, \tag{2.115}$$
$$d\Omega \wedge \Omega = 0\,, \tag{2.116}$$
$$H = -e^{2\varphi} * d(e^{-2\varphi}\Omega)\,, \tag{2.117}$$

where the H denotes the 3-form flux which is closed form and Φ denotes dilaton. If we identify the flux H the 3-form with the torsion T in (2.114) with $dT = 0$ and also dilaton φ with the Lee form Θ by $2d\varphi = \Theta$, (2.112), (2.113), and (2.114) are equivalent to (2.115), (2.116), and (2.117). The equations (2.115), (2.116), and (2.117) are equivalent to supersymmetry preserving conditions of the theory [24, 33].

2.4 $Spin(7)$-structures

A $Spin(7)$-structure $\mathcal{Q}_8$ is a principal subbundle of a frame bundle $\mathcal{L}(M_8)$ over an eight-dimensional real manifold M_8. A $Spin(7)$-structure is determined by the choice of a fundamental 4-form Ψ, which is satisfies the equation

$$*\Psi = \Psi\,. \tag{2.118}$$

The subgroup $Spin(7)$ is defined by a stabilizer subgroup of Ψ. Then, the structure $\mathcal{Q}_8$ has a structure group $Spin(7)$ and the group is a reduced subgroup of $SO(8)$, which is a structure group $\mathcal{L}(M_8)$. We decompose the Lie algebra $\mathfrak{so}(8)$ into $\mathfrak{spin}(7)$ and the orthogonal complement $\mathfrak{m}$, $\mathfrak{so}(8) = \mathfrak{spin}(7) \oplus \mathfrak{m}$. A defect of the holonomy group by contracting from $SO(8)$ to $Spin(7)$ is measured by an intrinsic torsion $\mathcal{T}$, which is identified with $\nabla\Psi$, where ∇ denotes Levi-Civitá connection. The torsion $\mathcal{T}$ belongs to $\boldsymbol{R}^8 \otimes \mathfrak{m}$ at local and the space of the intrinsic torsion is decomposed into two irreducible components by irreducible representations of $Spin(7)$,

$$\boldsymbol{R}^8 \otimes \mathfrak{m} = \mathcal{W}_1 \oplus \mathcal{W}_2, \tag{2.119}$$

where $dim\mathcal{W}_1 = 48$ and $dim\mathcal{W}_2 = 8$ [18] (see also Ref. [30]). Since $\mathcal{T}$ belongs to $\boldsymbol{R}^8 \otimes \mathfrak{m}$, we have

$$\nabla\Psi = \varrho(\mathcal{T})\Psi \,\in\, \mathcal{W}_1 \oplus \mathcal{W}_2\,. \tag{2.120}$$

According to Ref. [30], the classification of $Spin(7)$ is the following:

Name	type	conditions
$Spin(7)$	$\{0\}$	$d\Psi = 0$
	$\mathcal{W}_1$	$\Theta = 0$
Locally conformally $Spin(7)$	$\mathcal{W}_2$	$d\Psi = \Theta \wedge \Psi$
	$\mathcal{W}_1 \oplus \mathcal{W}_2$	no relations

Here Θ is a Lee form and this is defined by

$$\Theta = -\frac{1}{7} * (*d\Psi \wedge \Psi)\,. \tag{2.121}$$

We will study the class $\mathcal{W}_1 \oplus \mathcal{W}_2$ because the class is associated with type II or heterotic supergravity theory.

We consider an eight-dimensional real manifold admitting the type $\mathcal{W}_1 \oplus \mathcal{W}_2$ $Spin(7)$-structure. The $Spin(7)$-structure is defined by the condition that the Lee form should be an exact form and the dilaton in the supergravity theory is identified with the Lee form,

$$2d\varphi = \frac{7}{6}\Theta\,. \tag{2.122}$$

The $Spin(7)$-structure admits the existence of an unique connection ∇^T such that $\nabla^T\Psi = 0$ and the torsion is given by[9]

$$d\Psi - \frac{7}{6}\Theta \wedge \Psi = - * T\,. \tag{2.123}$$

Then, the geometry preserves a single Killing spinor which satisfies supersymmetric equations in the theory.

Chapter 3

Heterotic solutions with G_2 and Spin(7) structures

By solving the defining equations for the G_2-structure in the class $\mathcal{W}_3 \oplus \mathcal{W}_4$, supersymmetric solutions of the form $\boldsymbol{R}^{1,2} \times M_7$ in type II supergravity theory were constructed[4], where the flux H and the dilaton Φ were identified with 3-form torsion and Lee form, respectively. Similar to these solutions, we will solve the defining equations of the G_2 or $Spin(7)$-structure in the associated class with Abelian heterotic supergravity theory to obtain supersymmetric solutions. Unlike type II supergravity theory, Abelian heterotic supergravity theory additionally has a $U(1)$ gauge field. However, the field strength of the $U(1)$ gauge field is determined algebraically. Thus we will use a similar method to derive the supersymmetric solutions in Abelian heterotic theory. This chapter is based on Ref. [2].

3.1 G_2-structure associated with supergravity

In this section, we review a G_2-structure related with 7-dimensional Abelian heterotic supergravity theory. We first explain Abelian heterotic supergravity theory shortly. Next, we introduce the G_2-structure and derive the fundamental equations describing supersymmetric equations.

3.1.1 Abelian heterotic supergravity

According to Ref. [10], Abelian heterotic supergravity theory is a closely related to the low energy effective theory with the α' expansion of heterotic string theory, i.e., heterotic supergravity theory, which is obtained by formal some operations [10]. This system consists of a metric g, dilaton Φ, $U(1)$ gauge field A and B field. Then the string frame Lagrangian is given by

$$\mathcal{L} = e^{-\Phi}\left(R * 1 + *d\Phi \wedge d\Phi - *F \wedge F - \frac{1}{2} * H \wedge H\right), \tag{3.1}$$

where R represents a scalar curvature and $F = dA$. The 3-form flux H is given by $H = dB + A \wedge dA$, which satisfies the Bianchi identity

$$dH = F \wedge F. \tag{3.2}$$

In addition, $*$ represents the Hodge dual operator associated with the metric g. The equations of motion are written as

$$
\begin{aligned}
R_{\mu\nu} &= -\nabla_\mu\nabla_\nu\Phi + F_\mu{}^\rho F_{\nu\rho} + \frac{1}{4}H_\mu{}^{\rho\sigma}H_{\nu\rho\sigma}\,, && (3.3)\\
\nabla^2 e^{-\Phi} &= \frac{1}{2}e^{-\Phi}F_{\mu\nu}F^{\mu\nu} + \frac{1}{6}e^{-\Phi}H_{\mu\nu\rho}H^{\mu\nu\rho}\,, && (3.4)\\
d(e^{-\Phi} * F) &= -e^{-\Phi} * H\wedge F\,, && (3.5)\\
d(e^{-\Phi} * H) &= 0 && (3.6)
\end{aligned}
$$

and the corresponding supersymmetric equations are

$$
\nabla^T_\mu\chi \equiv \left(\nabla_\mu + \frac{1}{4\cdot 2!}H_{\mu\nu\rho}\gamma^{\nu\rho}\right)\chi = 0\,, \tag{3.7}
$$

$$
(\partial_\mu\Phi)\gamma^\mu\chi + \frac{1}{3!}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\chi = 0\,, \tag{3.8}
$$

$$
F_{\mu\nu}\gamma^{\mu\nu}\chi = 0\,, \tag{3.9}
$$

where χ denotes a Killing spinor. These equations (3.7), (3.8) and (3.9) are derived from the supersymmetry variation of gravitino, dilatino and gaugino, respectively.

It is known that if (g, Φ, H, F) satisfies the supersymmetric equations and the Bianchi identity, the quartet automatically solves the equations of motion [4, 23].In this chapter, we shall construct supersymmetric solutions in the Abelian heterotic theory. For this purpose, we consider 7- and 8-dimensional manifolds M_n admitting G_2 and $Spin(7)$ structures [1]. The corresponding 10-dimensional spacetimes are assumed to be of the form $\boldsymbol{R}^{1,9-n}\times M_n$ ($n = 7, 8$) where the fields are non-trivial only on M_n.

3.1.2 G_2-structure

A G_2-structure over M_7 is a principal subbundle with fiber G_2 of the frame bundle $\mathcal{F}(M_7)$ [6]. There is a one to one correspondence between G_2 structures and G_2 invariant 3-forms Ω. The standard form of Ω is given by

$$
\Omega = e^{123} + e^{147} + e^{165} + e^{246} + e^{257} + e^{354} + e^{367}\,, \tag{3.10}
$$

where e^μ ($\mu = 1, \cdots, 7$) is an orthonormal frame of the metric, $g = \sum_{\mu=1}^{7} e^\mu\otimes e^\mu$ and $e^{\mu\nu\rho} = e^\mu\wedge e^\nu\wedge e^\rho$.

We consider the G_2-structure Ω associated with heterotic supergravity. According to [24], we require

$$
d * \Omega = \Theta\wedge *\Omega\,, \tag{3.11}
$$

[1]For 6-dimensional case, supersymmetric solutions of Abelian heterotic theory were constructed in the view of supergravity solution generating method [10].

where Θ is the Lee form defined by

$$\Theta = -\frac{1}{3} * (*d\Omega \wedge \Omega) . \tag{3.12}$$

Then we have a unique connection ∇^T preserving the G_2-structure with 3-form torsion[23, 24, 34]

$$T = \frac{1}{6} * (d\Omega \wedge \Omega)\Omega - * (d\Omega - \Theta \wedge \Omega) \ , \tag{3.13}$$

i.e., $\nabla^T \Omega = \nabla^T g = 0$. This structure belongs to the class $\mathcal{W}_1 \oplus \mathcal{W}_3 \oplus \mathcal{W}_4$ in the classification by Fernández and Gray [13](see also [30]). It was shown in the papers [24], [34] that there exists a non-trivial solution to both dilatino and gravitino Killing spinor equations in dimension 7 if and only if there exists a G_2-structure satisfying (3.11) and

$$d\Omega \wedge \Omega \quad = \quad 0 \tag{3.14}$$

together with an exact Lee form Θ. Here (3.14) yields that this G_2-structure is in the class $\mathcal{W}_3 \oplus \mathcal{W}_4$. If $\Theta = 0$ and $T = 0$ one can find $d\Omega = 0$ and $d * \Omega = 0$, which are equivalent to $\nabla\Omega = 0$ and the corresponding metric is a Ricci-flat metric with G_2 holonomy. Now, we have the following identification

$$H = T \, , \tag{3.15}$$

$$d\Phi = \Theta \, . \tag{3.16}$$

Thus we obtain

$$H = -e^{\Phi} * d(e^{-\Phi}\Omega) \, , \quad d\Phi = -\frac{1}{3} * (*d\Omega \wedge \Omega) \, , \tag{3.17}$$

which means vanishing of supersymmetry variations for gravitino and dilatino, respectively. In addition, the supersymmetry variation for gaugino requires the generalized self-dual condition on the field strength F,

$$*(\Omega \wedge F) = F \, . \tag{3.18}$$

In general, 2-forms on M_7 are decomposed into G_2 irreducible representations under action of G_2,

$$\wedge^2_7 \quad = \quad \{\alpha \in \wedge^2 \, : \, *(\Omega \wedge \alpha) = -2\alpha\} \, , \tag{3.19}$$

$$\wedge^2_{14} \quad = \quad \{\alpha \in \wedge^2 \, : \, *(\Omega \wedge \alpha) = \alpha\} \, , \tag{3.20}$$

where subscripts $7, 14$ of $\wedge^2$ are dimensions of the representation. Thus the field strength F in (3.18) is in $\wedge^2_{14}$. The Bianchi identity (3.2) gives a relation between T and F, which leads to a strong restriction for the field strength F.

In summary, supersymmetric equations associated with Abelian heterotic supergravity theory are given by

$$
\begin{aligned}
& e^{\Phi} d(e^{-\Phi} * \Omega) = 0\,, && (3.21)\\
& d\Omega \wedge \Omega = 0\,, && (3.22)\\
& H = -e^{\Phi} * d(e^{-\Phi}\Omega)\,, && (3.23)\\
& d\Phi = -\frac{1}{3} * (*d\Omega \wedge \Omega)\,, && (3.24)\\
& *(\Omega \wedge F) = F && (3.25)
\end{aligned}
$$

together with the Bianchi identity

$$dH = F \wedge F\,. \tag{3.26}$$

The first equation is derived from (3.11) and (3.16) easily. In the following we call these equations G_2 with torsion equations or simply G_2T equations. Thus it is enough to solve G_2T equations so as to construct supersymmetric solutions.

3.2 G_2 solutions in 7-dimensional Abelian heterotic supergravity

In this section, we study G_2T equations under cohomogeneity one ansatz. Then these equations are reduced to first order non-linear ordinary differential equations. We solve reduced equations numerically and also try to construct analytically solutions for these equations.

3.2.1 Cohomogeneity one ansatz

The G_2T equations can be reduced to ordinary differential equations under cohomogeneity one ansatz. A G-manifold is called cohomogeneity one manifold if the principal orbits are hypersurfaces. It is known that cohomogeneity one manifolds admitting G_2-structures are classified into seven types [27]. In this paper we will consider the case when the principal orbits are $S^3 \times S^3$. Then the manifold is locally $\boldsymbol{R}_+ \times S^3 \times S^3$. Similar analysis can be done for the other orbits.

We consider the following metric with six radial functions $a_i(t), b_i(t)$ $(i = 1, 2, 3)$

$$g = dt^2 + \sum_{i=1}^{3} a_i(t)^2 (\sigma_i - \Sigma_i)^2 + \sum_{i=1}^{3} b_i(t)^2 (\sigma_i + \Sigma_i)^2\,, \tag{3.27}$$

where σ_i and Σ_i are left invariant 1-forms on two SU(2) group manifolds which satisfy the relations

$$
\begin{aligned}
& d\sigma_1 = -\sigma_2 \wedge \sigma_3\ ,\ d\sigma_2 = -\sigma_3 \wedge \sigma_1\ ,\ d\sigma_3 = -\sigma_1 \wedge \sigma_2\,,\\
& d\Sigma_1 = -\Sigma_2 \wedge \Sigma_3\ ,\ d\Sigma_2 = -\Sigma_3 \wedge \Sigma_1\ ,\ d\Sigma_3 = -\Sigma_1 \wedge \Sigma_2\,.
\end{aligned} \tag{3.28}
$$

It is convenient to introduce an orthonormal frame e^μ $(\mu = 1, \cdots, 7)$ defined by

$$
\begin{aligned}
& e^1 = a_1(t)(\sigma_1 - \Sigma_1)\ ,\ e^2 = a_2(t)(\sigma_2 - \Sigma_2)\ ,\ e^3 = a_3(t)(\sigma_3 - \Sigma_3)\,,\\
& e^4 = b_1(t)(\sigma_1 + \Sigma_1)\ ,\ e^5 = b_2(t)(\sigma_2 + \Sigma_2)\ ,\ e^6 = b_3(t)(\sigma_3 + \Sigma_3)\ ,\ e^7 = dt\,.
\end{aligned} \tag{3.29}
$$

Then the G_2-structure Ω takes the form (3.10), which automatically satisfy (3.22). The dilaton Φ is explicitly calculated from (3.24),

$$
\begin{aligned}
\frac{d\Phi}{dt} &= -\frac{1}{3}\Big(-\frac{2}{a_1}\frac{da_1}{dt}-\frac{2}{a_2}\frac{da_2}{dt}-\frac{2}{a_3}\frac{da_3}{dt}-\frac{2}{b_1}\frac{db_1}{dt}-\frac{2}{b_2}\frac{db_2}{dt}-\frac{2}{b_3}\frac{db_3}{dt}\\
&\quad +\frac{a_1}{2a_2b_3}+\frac{a_1}{2a_3b_2}+\frac{a_2}{2a_3b_1}+\frac{a_2}{2a_1b_3}+\frac{a_3}{2a_1b_2}+\frac{a_3}{2a_2b_1}\\
&\quad +\frac{b_1}{2a_2a_3}-\frac{b_1}{2b_2b_3}+\frac{b_2}{2a_1a_3}-\frac{b_2}{2b_1b_3}+\frac{b_3}{2a_1a_2}-\frac{b_3}{2b_1b_2}\Big)\,. \qquad (3.30)
\end{aligned}
$$

The 3-form flux $H=H_{126}e^{126}+H_{456}e^{456}+H_{135}e^{135}+H_{234}e^{234}$ is also calculated from (3.23),

$$
\begin{aligned}
H_{126} &= \frac{1}{a_3}\frac{da_3}{dt}+\frac{1}{b_1}\frac{db_1}{dt}+\frac{1}{b_2}\frac{db_2}{dt}-\frac{a_1}{2a_3b_2}-\frac{a_2}{2a_3b_1}+\frac{b_3}{2b_1b_2}-\frac{d\Phi}{dt}\,,\\
H_{456} &= -\frac{1}{a_1}\frac{da_1}{dt}-\frac{1}{a_2}\frac{da_2}{dt}-\frac{1}{a_3}\frac{da_3}{dt}+\frac{b_1}{2a_2a_3}+\frac{b_2}{2a_1a_3}+\frac{b_3}{2a_1a_2}+\frac{d\Phi}{dt}\,,\\
H_{135} &= -\frac{1}{a_2}\frac{da_2}{dt}-\frac{1}{b_1}\frac{db_1}{dt}-\frac{1}{b_3}\frac{db_3}{dt}+\frac{a_1}{2a_2b_3}-\frac{b_2}{2b_1b_3}+\frac{a_3}{2a_2b_1}+\frac{d\Phi}{dt}\,,\\
H_{234} &= \frac{1}{a_1}\frac{da_1}{dt}+\frac{1}{b_2}\frac{db_2}{dt}+\frac{1}{b_3}\frac{db_3}{dt}+\frac{b_1}{2b_2b_3}-\frac{a_2}{2a_1b_3}-\frac{a_3}{2a_1b_2}-\frac{d\Phi}{dt}\,, \qquad (3.31)
\end{aligned}
$$

which clearly depend on only t.

One may think that the ansatz (3.27) is artificial. However the ansatz (3.27) is reasonable from the view of a half-flat structure. A G_2 structure and 7-dimensional metric on $\boldsymbol{R}_+\times S^3\times S^3$ are given by a one-parameter family of a half-flat structure on $S^3\times S^3$ [35]. A half-flat structure is an $SU(3)$-structure, which is characterized by a pair (κ,γ), where κ is a symplectic form on $S^3\times S^3$ and γ is a real 3-form whose stabiliser is isomorphic to $SL(3;\boldsymbol{C})$ and satisfies a condition $\gamma\wedge\kappa=0$. If (κ,γ) satisfies the defining equation

$$
d(\kappa\wedge\kappa)=0\ ,\ d\gamma=0, \qquad (3.32)
$$

then the $SU(3)$-structure is called a half-flat structure[25]. Indeed the following half-flat pair (κ,γ),

$$
\begin{aligned}
\kappa &= p_1\sigma_1\wedge\Sigma_1+p_2\sigma_2\wedge\Sigma_2+p_3\sigma_3\wedge\Sigma_3\,, \qquad (3.33)\\
\gamma &= q_1\sigma_1\wedge\sigma_2\wedge\sigma_3+q_2\Sigma_1\wedge\Sigma_2\wedge\Sigma_3+q_3(\sigma_2\wedge\sigma_3\wedge\Sigma_1-\sigma_1\wedge\Sigma_2\wedge\Sigma_3)\\
&\quad +q_4(\sigma_3\wedge\sigma_1\wedge\Sigma_1-\sigma_2\wedge\Sigma_3\wedge\Sigma_1)+q_5(\sigma_1\wedge\sigma_2\wedge\Sigma_3-\sigma_3\wedge\Sigma_1\wedge\Sigma_2) \qquad (3.34)
\end{aligned}
$$

leads to the six dimensional part of the metric (3.27) under certain parameters p_i and q_j.

The field strength F is restricted by (3.25) and (3.26). Actually, we find F is classified into seven types,

$$
\begin{aligned}
&F_{12}e^{12}+F_{45}e^{45}+F_{67}e^{67}\ ,\ F_{13}e^{13}+F_{46}e^{46}+F_{57}e^{57}\ ,\ F_{23}e^{23}+F_{47}e^{47}+F_{56}e^{56}\\
&F_{14}e^{14}+F_{25}e^{25}+F_{36}e^{36}\ ,\ F_{15}e^{15}+F_{24}e^{24}+F_{37}e^{37}\ ,\ F_{16}e^{16}+F_{27}e^{27}+F_{34}e^{34}\,,\\
&F_{17}e^{17}+F_{26}e^{26}+F_{35}e^{35}\,, \qquad (3.35)
\end{aligned}
$$

together with generalized self-dual relations

$$
\begin{aligned}
&F_{13} - F_{46} = F_{57}\ ,\ F_{56} - F_{23} = F_{47}\ ,\ F_{45} - F_{12} = F_{67}\ ,\ F_{36} - F_{14} = F_{25} \\
&F_{24} - F_{15} = F_{37}\ ,\ F_{16} - F_{27} = F_{34}\ ,\ F_{35} - F_{26} = F_{17}\,.
\end{aligned}
\tag{3.36}
$$

Let us restrict ourselves to the case

$$
F = F_{12}e^{12} + F_{45}e^{45} + F_{67}e^{67}\,. \tag{3.37}
$$

Without loss of generality this case can be selected because other cases are obtained by using discrete symmetries of G_2T equations [2]. For examaple, G_2T equations are invariant under the following transformation,

$$
\begin{aligned}
&F_{12} \longrightarrow F_{23}\ ,\ F_{45} \longrightarrow F_{56}\ ,\ F_{67} \longrightarrow F_{47}\,, \\
&a_1 \longrightarrow a_2\ ,\ a_2 \longrightarrow a_3\ ,\ a_3 \longrightarrow a_1\ ,\ b_1 \longrightarrow b_2\ ,\ b_2 \longrightarrow b_3\ ,\ b_3 \longrightarrow b_1\,, \\
&H_{126} \longrightarrow H_{234}\ ,\ H_{135} \longrightarrow -H_{126}\ ,\ H_{456} \longrightarrow H_{456}\ ,\ H_{234} \longrightarrow -H_{135}\,.
\end{aligned}
\tag{3.38}
$$

We have already written down (3.22), (3.23), (3.24) and (3.25) under the cohomogeneity one ansatz. The remaining G_2T equation we should solve are (3.21) and (3.26). The former is given by

$$
\begin{aligned}
\frac{1}{a_1}\frac{da_1}{dt} + \frac{1}{a_2}\frac{da_2}{dt} + \frac{1}{b_1}\frac{db_1}{dt} + \frac{1}{b_2}\frac{db_2}{dt} - \frac{b_3}{2a_1a_2} - \frac{a_3}{2a_1b_2} - \frac{a_3}{2a_2b_1} + \frac{b_3}{2b_1b_2} - \frac{d\Phi}{dt} &= 0\,, \\
\frac{1}{a_1}\frac{da_1}{dt} + \frac{1}{a_3}\frac{da_3}{dt} + \frac{1}{b_1}\frac{db_1}{dt} + \frac{1}{b_3}\frac{db_3}{dt} - \frac{b_2}{2a_1a_3} - \frac{a_2}{2a_1b_3} + \frac{b_2}{2b_1b_3} - \frac{a_2}{2a_3b_1} - \frac{d\Phi}{dt} &= 0\,, \\
\frac{1}{a_2}\frac{da_2}{dt} + \frac{1}{a_3}\frac{da_3}{dt} + \frac{1}{b_2}\frac{db_2}{dt} + \frac{1}{b_3}\frac{db_3}{dt} - \frac{b_1}{2a_2a_3} + \frac{b_1}{2b_2b_3} - \frac{a_1}{2a_2b_3} - \frac{a_1}{2a_3b_2} - \frac{d\Phi}{dt} &= 0\,,
\end{aligned}
\tag{3.39}
$$

and the latter is given by

$$
\begin{aligned}
\frac{dH_{456}}{dt} + H_{456}\left(\frac{1}{b_1}\frac{db_1}{dt} + \frac{1}{b_2}\frac{db_2}{dt} + \frac{1}{b_3}\frac{db_3}{dt}\right) &= -2F_{45}F_{67}\,, \\
\frac{dH_{126}}{dt} + H_{126}\left(\frac{1}{a_1}\frac{da_1}{dt} + \frac{1}{a_2}\frac{da_2}{dt} + \frac{1}{b_3}\frac{db_3}{dt}\right) &= -2F_{12}F_{67}\,, \\
\frac{dH_{135}}{dt} + H_{135}\left(\frac{1}{a_1}\frac{da_1}{dt} + \frac{1}{a_3}\frac{da_3}{dt} + \frac{1}{b_2}\frac{db_2}{dt}\right) &= 0\,, \\
\frac{dH_{234}}{dt} + H_{234}\left(\frac{1}{a_2}\frac{da_2}{dt} + \frac{1}{a_3}\frac{da_3}{dt} + \frac{1}{b_1}\frac{db_1}{dt}\right) &= 0\,,
\end{aligned}
\tag{3.40}
$$

[2]The field strength $F = F_{14}e^{14} + F_{25}e^{25} + F_{36}e^{36}$ is exceptional. In this case, G_2T equations lead to the solution $F = 0$, which was studied in the paper[4].

$$
\begin{aligned}
H_{126}\frac{a_1}{2a_3b_2} - H_{456}\frac{b_1}{2a_2a_3} - H_{135}\frac{a_1}{2a_2b_3} - H_{234}\frac{b_1}{2b_2b_3} &= 0\,,\\
H_{126}\frac{a_2}{2a_3b_1} - H_{456}\frac{b_2}{2a_1a_3} + H_{135}\frac{b_2}{2b_1b_3} + H_{234}\frac{a_2}{2a_1b_3} &= 0\,,\\
H_{126}\frac{b_3}{2b_1b_2} + H_{456}\frac{b_3}{2a_1a_2} + H_{135}\frac{a_3}{2a_2b_1} - H_{234}\frac{a_3}{2a_1b_2} &= -2F_{12}F_{45}\,,
\end{aligned}
$$

which consist of differential equations (3.40) and algebraic equations (3.41). The algebraic equations (3.41) yields the following relations,

$$
\begin{aligned}
H_{456} &= \frac{a_1a_2}{b_1b_2}H_{126}\ ,\ H_{234} = -\frac{a_1b_2}{a_2b_1}H_{135}\,,\\
F_{12} &= \pm\sqrt{-\frac{1}{2}\left(\frac{b_3}{a_1a_2}H_{126} + \frac{a_3b_2}{a_1{a_2}^2}H_{135}\right)}\,,\\
F_{45} &= \frac{a_1a_2}{b_1b_2}F_{12}\ ,\ F_{67} = F_{45} - F_{12}
\end{aligned}
\tag{3.41}
$$

and a_i, b_i, H_{126} and H_{135} are determined by the differential equations (3.31), (3.39) and (3.40) :

$$
\frac{da_1}{dt} = \frac{-{a_1}^2 + {a_2}^2 + {b_3}^2}{4a_2b_3} + \frac{-{a_1}^2 + {a_3}^2 + {b_2}^2}{4a_3b_2} - \frac{1}{2}a_1H_{126} + \frac{1}{2}a_1H_{135}\,, \tag{3.42}
$$

$$
\frac{da_2}{dt} = \frac{-{a_2}^2 + {a_1}^2 + {b_3}^2}{4a_1b_3} + \frac{-{a_2}^2 + {a_3}^2 + {b_1}^2}{4a_3b_1} - \frac{1}{2}a_2H_{126} - \frac{1}{2}a_2H_{234}\,, \tag{3.43}
$$

$$
\frac{da_3}{dt} = \frac{-{a_3}^2 + {a_2}^2 + {b_1}^2}{4a_2b_1} + \frac{-{a_3}^2 + {a_1}^2 + {b_2}^2}{4a_1b_2} + \frac{1}{2}a_3H_{135} - \frac{1}{2}a_3H_{234}\,, \tag{3.44}
$$

$$
\frac{db_1}{dt} = \frac{-{b_1}^2 + {a_2}^2 + {a_3}^2}{4a_2a_3} + \frac{{b_1}^2 - {b_2}^2 - {b_3}^2}{4b_2b_3} + \frac{1}{2}b_1H_{456} - \frac{1}{2}b_1H_{234}\,, \tag{3.45}
$$

$$
\frac{db_2}{dt} = \frac{-{b_2}^2 + {a_1}^2 + {a_3}^2}{4a_3a_1} + \frac{{b_2}^2 - {b_1}^2 - {b_3}^2}{4b_3b_1} + \frac{1}{2}b_2H_{456} + \frac{1}{2}b_2H_{135}\,, \tag{3.46}
$$

$$
\frac{db_3}{dt} = \frac{-{b_3}^2 + {a_1}^2 + {a_2}^2}{4a_1a_2} + \frac{{b_3}^2 - {b_1}^2 - {b_2}^2}{4b_1b_2} - \frac{1}{2}b_3H_{126} + \frac{1}{2}b_3H_{456}\,, \tag{3.47}
$$

$$
\frac{dH_{126}}{dt} + H_{126}\left(\frac{1}{a_1}\frac{da_1}{dt} + \frac{1}{a_2}\frac{da_2}{dt} + \frac{1}{b_3}\frac{db_3}{dt}\right) = \left(\frac{a_1a_2}{b_1b_2} - 1\right)\left(\frac{b_3}{a_1a_2}H_{126} + \frac{a_3b_2}{a_1{a_2}^2}H_{135}\right)\,, \tag{3.48}
$$

$$
\frac{dH_{135}}{dt} + H_{135}\left(\frac{1}{a_1}\frac{da_1}{dt} + \frac{1}{a_3}\frac{da_3}{dt} + \frac{1}{b_2}\frac{db_2}{dt}\right) = 0\,. \tag{3.49}
$$

Thus we have obtained G_2T equations (3.42)–(3.49) under the cohomogeneity one ansatz with principal orbits $S^3 \times S^3$. It should be noticed that dilaton is given by (3.30), which is equivalent to

$$
\frac{d\Phi}{dt} = -H_{126} + H_{456} + H_{135} - H_{234}\,. \tag{3.50}
$$

Note that it is automatically assured by (3.41)–(3.49) that the field strength F is closed two-form, $dF = 0$.

3.2.2 G_2T solutions

In order to construct regular metrics satisfying (3.42)–(3.49), we shall look for solutions with bolt singularities at $t = 0$. There exist two types of bolt singularity corresponding to a collapsing S^3 or S^1 at $t = 0$. In the torsion free case, i.e. $H_{\mu\nu\rho} = 0$, these equations were studied by Cvetič, Gibbons, Lü and Pope [16][17] and they found Ricci-flat metrics with G_2 holonomy. These equations are also obtained from $d\Omega = 0$ and $d * \Omega = 0$. Let us turn to non-zero flux. Although (3.42)–(3.49) admit a Taylor expansion around $t = 0$, numerical analysis shows that global solution doesn't exist for general six radial functions $a_i(t), b_i(t)$. Hence we study the reduced case,

$$a_1(t) = a_2(t) \equiv a(t) \ , \ b_1(t) = b_2(t) \equiv b(t). \tag{3.51}$$

Note that the Taylor expansion around $t = 0$ gives rise to $H_{135} = 0$.

By (3.51) and $H_{135} = 0$ the reduced system is given by

$$\frac{1}{a}\frac{da}{dt} = \frac{b}{4aa_3} + \frac{a_3}{4ab} - \frac{a}{4a_3 b} + \frac{b_3}{4a^2} - \frac{1}{2}H_{126}\,, \tag{3.52}$$

$$\frac{1}{b}\frac{db}{dt} = -\frac{b}{4aa_3} + \frac{a_3}{4ab} + \frac{a}{4a_3 b} - \frac{b_3}{4b^2} + \frac{a^2}{2b^2}H_{126}\,, \tag{3.53}$$

$$\frac{1}{a_3}\frac{da_3}{dt} = \frac{b}{2aa_3} - \frac{a_3}{2ab} + \frac{a}{2a_3 b}\,, \tag{3.54}$$

$$\frac{1}{b_3}\frac{db_3}{dt} = -\frac{b_3}{4a^2} + \frac{b_3}{4b^2} + \frac{1}{2}\left(\frac{a^2}{b^2} - 1\right)H_{126}\,, \tag{3.55}$$

$$\frac{dH_{126}}{dt} = -H_{126}\left[\frac{b^2 + a_3^2 - a^2}{2aa_3 b} + \frac{b_3(5b^2 - 3a^2)}{4a^2 b^2}\right] - \frac{1}{2}\left(\frac{a^2}{b^2} - 3\right)H_{126}^2\,, \tag{3.56}$$

$$\frac{d\Phi}{dt} = H_{126}\left(\frac{a^2}{b^2} - 1\right)\,. \tag{3.57}$$

Then, the algebraic equations (3.41) are written as

$$\begin{aligned} H_{456} &= \frac{a^2}{b^2}H_{126}\,, \\ F_{12} &= \pm\sqrt{-\frac{1}{2}\left(\frac{b_3}{a^2}H_{126}\right)}\,, \\ F_{45} &= \frac{a^2}{b^2}F_{12}\,, \\ F_{67} &= F_{45} - F_{12}\,. \end{aligned} \tag{3.58}$$

Now let us solve the equations (3.52)–(3.57). We put the form

$$b_3(t) = \frac{1}{h(t)}\beta_3(t)\,, \tag{3.59}$$

where the radial function $\beta_3(t)$ is defined as the solution to the ensuing equations,

$$\begin{aligned}
\frac{1}{a}\frac{da}{dt} &= \frac{b}{4aa_3}+\frac{a_3}{4ab}-\frac{a}{4a_3b}+\frac{\beta_3}{4a^2}\,, &(3.60)\\
\frac{1}{b}\frac{db}{dt} &= -\frac{b}{4aa_3}+\frac{a_3}{4ab}+\frac{a}{4a_3b}-\frac{\beta_3}{4b^2}\,, &(3.61)\\
\frac{1}{a_3}\frac{da_3}{dt} &= \frac{b}{2aa_3}-\frac{a_3}{2ab}+\frac{a}{2a_3b}\,, &(3.62)\\
\frac{1}{\beta_3}\frac{d\beta_3}{dt} &= -\frac{\beta_3}{4a^2}+\frac{\beta_3}{4b^2}\,, &(3.63)
\end{aligned}$$

which are obtained by setting $H_{126}=0$ in (3.52)–(3.57). The function $h(t)$ is determined as follows. From (3.52)–(3.55), (3.59)–(3.63), we find the relation

$$\frac{1}{h}\frac{dh}{dt}=-\frac{d\Phi}{dt}\,, \tag{3.64}$$

which yields

$$\frac{1}{\beta_3}\frac{d\beta_3}{dt}=\frac{1}{2(h-1)}\frac{dh}{dt}\,. \tag{3.65}$$

Thus we have

$$h(t)=k\beta_3(t)^2+1\,. \tag{3.66}$$

Then the relation (3.59) is given by

$$b_3(t)=\frac{1}{k\beta_3(t)^2+1}\beta_3(t) \tag{3.67}$$

and using (3.57), (3.64) and (3.66) leads to

$$H_{126}=-\frac{k{\beta_3}^3}{2a^2(k\beta_3^2+1)}\,. \tag{3.68}$$

For the positive constant k, the algebraic equations (3.58) gives H_{456} and $F_{\mu\nu}$,

$$\begin{aligned}
&H_{456}=-\frac{k{\beta_3}^3}{2b^2(k\beta_3^2+1)}\,,\ F_{12}=\pm\frac{\sqrt{k}{\beta_3}^2}{2a^2(k{\beta_3}^2+1)}\,,\\
&F_{45}=\pm\frac{\sqrt{k}{\beta_3}^2}{2b^2(k{\beta_3}^2+1)}\,,\ F_{67}=\pm\frac{\sqrt{k}{\beta_3}^2}{2(k{\beta_3}^2+1)}\left(-\frac{1}{a^2}+\frac{1}{b^2}\right)\,.
\end{aligned} \tag{3.69}$$

Thus, under the ansatz (3.59), solutions to (3.52)–(3.57) and (3.58) are written as

$$\begin{aligned}
g &= dt^2+a(t)^2\{(\sigma_1-\Sigma_1)^2+(\sigma_2-\Sigma_2)^2\}+a_3(t)^2(\sigma_3-\Sigma_3)^2\\
&\quad +b(t)^2\{(\sigma_1+\Sigma_1)^2+(\sigma_2+\Sigma_2)^2\}+\frac{1}{(k\beta_3(t)^2+1)^2}\beta_3(t)^2(\sigma_3+\Sigma_3)^2\,,
\end{aligned} \tag{3.70}$$

$$\Phi(t)=\log\frac{m}{k\beta_3(t)^2+1}\,, \tag{3.71}$$

$$H=-\frac{k\beta_3(t)^3}{2a(t)^2(k\beta_3(t)^2+1)}e^{126}-\frac{k\beta_3(t)^3}{2b(t)^2(k\beta_3(t)^2+1)}e^{456}\,, \tag{3.72}$$

$$F=\frac{\sqrt{k}\beta_3{}^2}{2a^2(k\beta_3{}^2+1)}e^{12}+\frac{\sqrt{k}\beta_3{}^2}{2b^2(k\beta_3{}^2+1)}e^{45}+\frac{\sqrt{k}\beta_3{}^2}{2(k\beta_3{}^2+1)}\left(-\frac{1}{a^2}+\frac{1}{b^2}\right)e^{67}\,, \tag{3.73}$$

where m is an arbitrary constant [3]. This formula includes additional one parameter m comparing to the formula obtained by $\mathrm{r}^{-1}\mathrm{Tr}$ transformation[10]. Now the problem of finding the solutions to G_2T equations reduced to that of finding the solutions to the torsion free equations(3.60)–(3.63). It should be noticed that the torsion free equations were investigated in the study of Ricci-flat metrics with G_2 holonomy [16]. It was proved[36] that there exist the solutions with a collapsing S^1 for the torsion free equations (3.60)–(3.63).

A regular solution to (3.60)–(3.63) representing a Ricci-flat metric with G_2 holonomy is known [5],

$$\begin{aligned} a_3(r) &= -\frac{1}{2}r\;,\; a(r)=\frac{1}{4}\sqrt{3(r-l)(r+3l)}\,, \\ \beta_3(r) &= l\sqrt{\frac{r^2-9l^2}{r^2-l^2}}\;,\; b(r)=-\frac{1}{4}\sqrt{3(r+l)(r-3l)}\,, \end{aligned} \tag{3.74}$$

where r is defined by $dt=\frac{3}{2}\frac{l}{\beta_3}dr$ and l is a scaling parameter ($l>0$). The corresponding solutions of G_2T equations are given by

$$\begin{aligned} g &= \frac{9}{4}\frac{r^2-l^2}{r^2-9l^2}dr^2+\frac{3}{16}(r-l)(r+3l)\{(\sigma_1-\Sigma_1)^2+(\sigma_2-\Sigma_2)^2\}+\frac{1}{4}r^2(\sigma_3-\Sigma_3)^2 \\ &\quad +\frac{3}{16}(r+l)(r-3l)\{(\sigma_1+\Sigma_1)^2+(\sigma_2+\Sigma_2)^2\}+\frac{l^2(r^2-9l^2)}{kl^2(r^2-9l^2)+r^2-l^2}(\sigma_3+\Sigma_3)^2\,, \\ H &= -\frac{8k}{3\left(1+k\frac{r^2-9l^2}{r^2-l^2}\right)}\sqrt{\frac{r^2-9l^2}{r^2-l^2}}\left[\frac{r-3l}{3(r-l)^2(r+l)}e^{126}+\frac{r+3l}{(r+l)^2(r-l)}e^{456}\right]\,, \\ F &= -\frac{8\sqrt{k}}{3\left(1+k\frac{r^2-9l^2}{r^2-l^2}\right)}\left[\frac{r-3l}{3(r-l)^2(r+l)}e^{12}+\frac{r+3l}{(r+l)^2(r-l)}e^{45}+\frac{2(r^2-3l^2)}{(r^2-l^2)^2}e^{67}\right]\,, \\ \Phi &= -m\log\left[1+k\frac{r^2-9l^2}{r^2-l^2}\right]\,. \end{aligned} \tag{3.75}$$

These solutions include two free parameters l and k [4]. When $k=0$, the solution reduces to the Ricci-flat metric with G_2 holonomy. The parameter r is larger than $3l$ by a regular condition and the other parameter k represents deformation from the Ricci-flat metric.

The metric (3.75) is an ALC(Asymptotically Locally Conical) metric and the Riemannian

[3] If we set $k=tan^2\delta$, $m=\frac{1}{cos^2\delta}$ and hence $m=k+1$, then these solutions are coincident with the solutions obtained by $\mathrm{r}^{-1}\mathrm{Tr}$ transformation in the paper [10]. When we take a limit $k\to\infty$, $m\to\infty$ keeping $\frac{k}{m}\to 1$, then δ is equal to $\frac{\pi}{2}$, which corresponds to T-dual transformation.

[4] For $m=k+1$, the equation (3.75) is also obtained by solution-generating method [10].

tensor is non-singular in the region $3l \leq r < \infty$. Further the scalar curvature R, $F^2 = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}$ and $H^2 = \frac{1}{6} H_{\mu\nu\rho} H^{\mu\nu\rho}$ are finite but their integrations diverge. The cause of the divergence comes from higher powers of r in the volume form, $\sqrt{g} \sim r^5$.

S^3-bolt solution

From now on, we explain the reason why we can put the relation (3.59). The boundary condition representing a collapsing S^3 at $t = 0$ imposes $b(t), b_3(t) \to 0$ for $t \to 0$. Indeed, the series solution around $t = 0$ takes the following form,

$$
\begin{aligned}
a(t) &= p_1 + \frac{1}{16p_1^2} t^2 - \frac{7 - 64p_1^2 p_2}{2560 p_1^3} t^4 + \cdots \,, \\
a_3(t) &= -p_1 - \frac{1}{16p_1^2} t^2 + \frac{6 + 128 p_1^2 p_2}{2560 p_1^3} t^4 + \cdots \,, \\
b(t) &= -\frac{1}{4} t + p_2 t^3 - \frac{1 + 1344 p_1^2 p_2 - 98304 p_1^4 p_2^2}{10240 p_1^4} t^5 + \cdots \,, \\
b_3(t) &= \frac{1}{4} t + \frac{128 p_1^2 p_2 + 128 p_1^4 p_3 - 1}{64 p_1^2} t^3 \\
&\quad - \frac{216 p_1^2 p_2 - 16896 p_1^4 p_2^2 + 240 p_1^4 p_3 - 30720 p_1^6 p_2 p_3 - 10240 p_1^8 p_3^2 - 1}{640 p_1^4} t^5 + \cdots \,, \\
H_{126}(t) &= p_3 t^3 + \left(8 p_1^2 p_3^2 + 24 p_2 p_3 - \frac{5 p_3}{16 p_1^2} \right) t^5 + \cdots \,,
\end{aligned} \tag{3.76}
$$

where p_1, p_2 and p_3 are free parameters. Then the metric behaves as

$$
\begin{aligned}
g \;\; \to \;\; & dt^2 + p_1^2 [(\sigma_1 - \Sigma_1)^2 + (\sigma_2 - \Sigma_2)^2 + (\sigma_3 - \Sigma_3)^2] \\
& + \frac{t^2}{16} [(\sigma_1 + \Sigma_1)^2 + (\sigma_2 + \Sigma_2)^2 + (\sigma_3 + \Sigma_3)^2] \,.
\end{aligned} \tag{3.77}
$$

Thus the metric has a removable bolt singularity (S^3-bolt solution). The parameter p_1 is a scaling parameter and $p_1 = \frac{3}{2} l$ for the solutions (3.75), while p_2 parametrizes a family of the regular solutions to (3.60)–(3.63). The remaining parameter p_3 is essentially the deformation parameter k associated with 3-form flux (3.68),

$$
p_3 = -\frac{k}{128 p_1^2} \,. \tag{3.78}
$$

Actually when $p_3 = 0$, the flux component $H_{126}(t)$ vanishes and the corresponding solution reduces to Ricci-flat metrics with a collapsing S^3 at $t = 0$ [16]. The radial functions $a(t)$, $b(t)$ and $a_3(t)$ are independent on p_3, which means that these functions are the same as those of the Ricci-flat metrics. On the other hand, $b_3(t)$ depends on p_3 so that this is different from $\beta_3(t)$ satisfying (3.60)–(3.63). Namely $b_3(t)$ is only deformed by the 3-form flux and hence we can take the form (3.59).

It should be noticed that we can obtain all S^3-bolt solutions of (3.52)–(3.57) from regular solutions of (3.60)–(3.63). In particular, the parameters $p_1 = \frac{3}{2}l$ and $p_2 = -\frac{1}{768p_1^2}$ correspond to the solutions (3.75). The numerical calculation requires the inequality of the parameter p_2 for each values of deformation parameter p_3 in order to extend the solution in large t region. For example when $p_1 = 1$ and $p_3 = -900$, the parameter p_2 is restricted by the inequality $-7.029 \times 10^3 \leq p_2 \leq 1.321 \times 10^{-3}$ (see Figure 3.1).

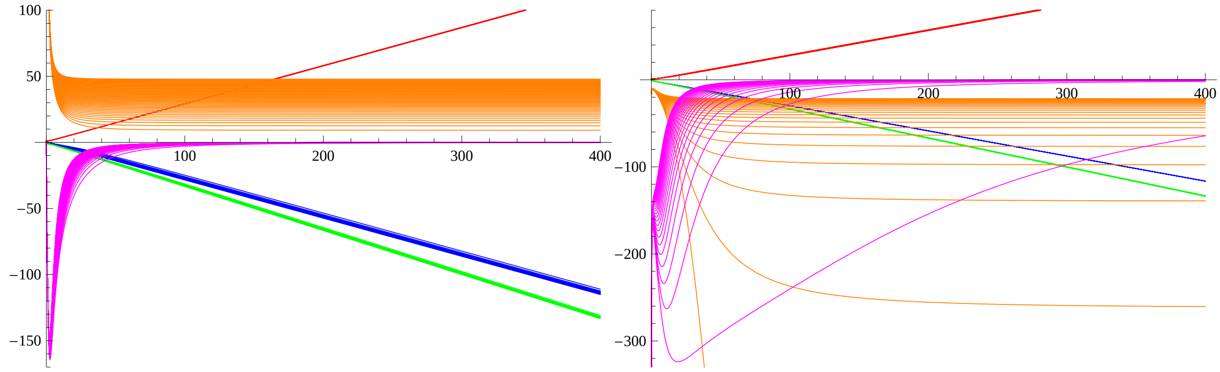

Figure 3.1: The both figures show S^3-bolt solutions. The left figure is the solution with $p_1 = 1$, $-1.321 \times 10^{-3} \leq p_2 \leq 1.321 \times 10^{-3}$ and $p_3 = -900$ and the right figure is the solution with $p_1 = 1$, $-7.029 \times 10^3 \leq p_2 \leq -6.852 \times 10^3$ and $p_3 = -900$. The red, green, blue, orange and magenta lines represent components $a(t)$, $a_3(t)$, $b(t)$, b_3 and $H_{126}(t)$, respectively.

$T^{1,1}$-bolt solution

Regular solutions with a collapsing S^1 takes the following form around $t = 0$,

$$
\begin{aligned}
a(t) &= p_1 + \frac{p_2 - 2p_1^2 p_3}{8p_1} t - \frac{3p_2^2 - 16p_1^2 - 12p_1^2 p_2 p_3 + 12p_1^4 p_3^2}{128p_1^3} t^2 + \cdots, \\
a_3(t) &= t - \frac{16p_1^2 - p_2^2 + 4p_1^2 p_2 p_3 - 4p_1^4 p_3^2}{96p_1^4} t^3 + \cdots, \\
b(t) &= p_1 - \frac{p_2 - 2p_1^2 p_3}{8p_1} t - \frac{3p_2^2 - 16p_1^2 - 12p_1^2 p_2 p_3 + 12p_1^4 p_3^2}{128p_1^3} t^2 + \cdots, \\
b_3(t) &= p_2 + \frac{4p_2^3 - 16p_2 p_3^2}{64p_1^4} t^2 + \cdots, \\
H_{126}(t) &= p_3 + \frac{2p_1^2 p_3^2 - p_2 p_3}{4p_1^2} t - \frac{-9p_2^2 p_3 + 8p_1^2 p_3 + 28p_1^2 p_2 p_3^2 - 20p_1^4 p_3^3}{32p_1^4} t^2 + \cdots, \quad (3.79)
\end{aligned}
$$

where p_1, p_2 and p_3 are free parameters. Then the metric behaves as

$$g \longrightarrow dt^2 + t^2(\sigma_3 - \Sigma_3)^2 + p_1^2(\sigma_1^2 + \Sigma_1^2 + \sigma_2^2 + \Sigma_2^2) + p_2^2(\sigma_3 + \Sigma_3)^2 \,. \tag{3.80}$$

When this metric has $T^{1,1}$-bolt, p_1 and p_2 have relation ([17])

$$p_2 = \pm\sqrt{\frac{2}{3}} p_1 \,. \tag{3.81}$$

In the series (3.79), all components are deformed by the 3-form flux, which implies that the relation like (3.59) doesn't exist in the solution. A numerical analysis indicates that the series solution (3.79) is extended to large t region (see Figure 3.2) [5].

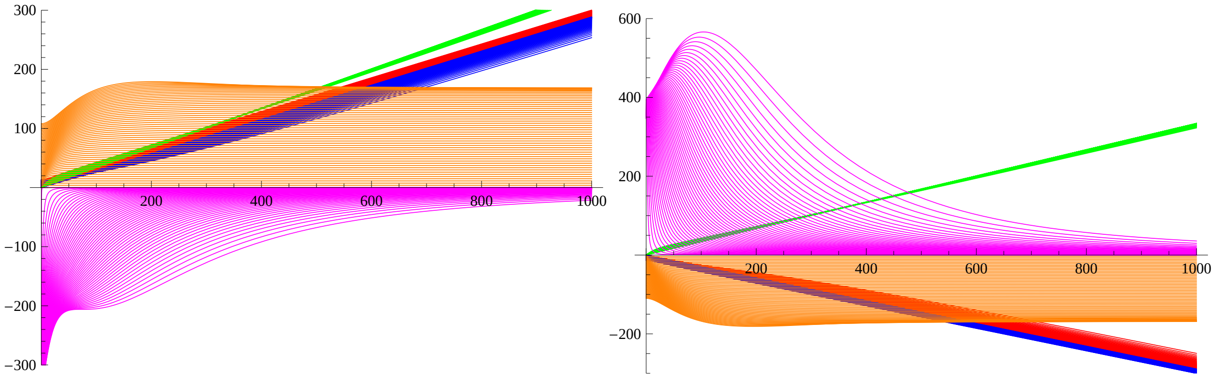

Figure 3.2: The left figure shows a $T^{1,1}$-bolt solution with $p_1 = \sqrt{\frac{3}{2}} p_2$, $0 < p_2 < 12$ and $p_3 = -\frac{1}{288}$. The right figure shows a $T^{1,1}$-bolt solution with $p_1 = \sqrt{\frac{3}{2}} p_2$, $-12 < p_2 < 0$ and $p_3 = \frac{1}{288}$. The red, green, blue, orange and magenta lines represent components $a(t)$, $a_3(t)$, $b(t)$, b_3 and $H_{126}(t)$, respectively.

3.3 $Spin(7)$ solutions in 8-dimensional Abelian heterotic supergravity

3.3.1 $Spin(7)$-structure

A $Spin(7)$-structure over M_8 is a principal subbundle with fiber $Spin(7)$ of the frame bundle $\mathcal{F}(M_8)$ [6]. There is a one to one correspondence between $Spin(7)$ structures and $Spin(7)$ invariant 4-forms Ψ. The standard form of Ψ is given by

$$\Psi = e^{8123} + e^{8145} + e^{8167} + e^{8264} + e^{8257} + e^{8347} + e^{8356} - e^{4567} - e^{2367} - e^{2345} - e^{3157} - e^{1346} - e^{1256} - e^{1247}, \tag{3.82}$$

[5] In the Ricci flat case, the existence of G_2 metrics with a collapsing S^1 was proved in [36].

where $\{e^\mu, \mu = 1, \cdots, 8\}$ is an orthonormal frame of the metric $g = \sum_{\mu=1}^{8} e^\mu \otimes e^\mu$. The 4-form Ψ is self-dual $*\Psi = \Psi$ for a volume form vol $= e^{12\cdots 8}$. The Lee form is defined by

$$\Theta = -\frac{1}{7} * (*d\Psi \wedge \Psi). \tag{3.83}$$

Then, there always exists a unique connection ∇^T preserving the $Spin(7)$ structure, $\nabla^T \Psi = \nabla^T g = 0$, with 3-form torsion [37]

$$T = *d\Psi - \frac{7}{6} * (\Theta \wedge \Psi). \tag{3.84}$$

This is different from the case of G_2, where we required an additional condition (3.11). However, except for this point we have very similar supersymmetry equations associated with Abelian heterotic supergravity. The Lee form is an exact 1-form, $\Theta = (6/7)d\Phi$, and hence after identification $T = H$ the equations are written as

$$H = e^{\Phi} * d(e^{-\Phi}\Psi)\,, \tag{3.85}$$

$$d\Phi = -\frac{1}{6} * (*d\Psi \wedge \Psi)\,, \tag{3.86}$$

$$*(\Psi \wedge F) = F\,, \tag{3.87}$$

$$dH = F \wedge F\,, \tag{3.88}$$

which correspond to G_2T equations. As with G_2, we call these equations $Spin(7)$ with torsion equations.

3.3.2 3-Sasakian ansatz

In order to construct explicit solutions of $Spin(7)$ with torsion equations, we will consider a special case that gives rise to ordinary differential equations and generalizes many known examples. Let us assume the following geometrical condition for 7-dimensional manifolds. Let $\pi : P \to B$ be a principal SO(3)-bundle over a compact self-dual Einstein manifold (or orbifold) B. Then, it is known that the total space P admits a 3-Sasakian structure. For details on 3-Sasakian structure we refer the reader to [38]. The connection 1-forms ϕ^i $(i = 1, 2, 3)$ will be chosen so that the corresponding curvature 2-forms

$$\omega^i = d\phi^i + \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}\phi^j \wedge \phi^k \tag{3.89}$$

are self-dual 2-forms on B,

$$\omega^1 = \theta^{45} + \theta^{67}, \quad \omega^2 = \theta^{64} + \theta^{57}, \quad \omega^3 = \theta^{47} + \theta^{56}, \tag{3.90}$$

which implies

$$\omega^1 \wedge \omega^2 = \omega^1 \wedge \omega^3 = \omega^2 \wedge \omega^3 = 0\ , \tag{3.91}$$

$$\omega^1 \wedge \omega^1 = \omega^2 \wedge \omega^2 = \omega^3 \wedge \omega^3 = 2vol_B\ , \tag{3.92}$$

$$\mathrm{vol}_B = \theta^{4567} \qquad \text{(volume form on } B\text{)} \tag{3.93}$$

and

$$
\begin{aligned}
d\omega^1 &= \omega^2 \wedge \phi^3 - \phi^2 \wedge \omega^3 \,, \\
d\omega^2 &= \omega^3 \wedge \phi^1 - \phi^3 \wedge \omega^1 \,, \\
d\omega^3 &= \omega^1 \wedge \phi^2 - \phi^1 \wedge \omega^2 \,,
\end{aligned}
\tag{3.94}
$$

On P the metric

$$
g_S = \sum_{i=1}^{3} \phi^i \otimes \phi^i + \sum_{a=4}^{7} \theta^a \otimes \theta^a
\tag{3.95}
$$

is a 3-Sasakian metric. Indeed, the Killing vector fields ξ_i $(i = 1, 2, 3)$ dual to the 1-forms ϕ^i give the characteristic vector fields of the 3- Sasakian structure satisfying the relations $[\xi_i,\, \xi_j] = \epsilon_{ijk}\, \xi_k$.

Now we consider a $Spin(7)$ structure on an 8-dimensional manifold $M = \boldsymbol{R}_+ \times P$. Using the 3-Sasakian structure we consider the following metric with four radial functions $\{a_i(t)\} = \{a(t),\, b(t),\, c(t)\}$ and $f(t)$:

$$
g = dt^2 + \sum_{i=1}^{3} a_i(t)^2 \phi^i \otimes \phi^i + f(t)^2 \sum_{a=4}^{7} \theta^a \otimes \theta^a \,.
\tag{3.96}
$$

For this metric an orthonormal frame e^μ $(\mu = 1, \cdots, 8)$ is introduced by

$$
e^i = a_i(t)\phi^i, \quad e^a = f(t)\theta^a, \quad e^8 = dt.
\tag{3.97}
$$

Then the 1-forms e^μ lead to a $Spin(7)$ structure Ψ on $M = \boldsymbol{R}_+ \times P$ given by (3.82). In terms of ϕ^i, ω^i and vol_B we have

$$
\begin{aligned}
\Psi \;\; = \;\; & abc\, dt \wedge \phi^1 \wedge \phi^2 \wedge \phi^3 - f^4 \mathrm{vol}_B + f^2 (a\, dt \wedge \phi^1 - bc\phi^2 \wedge \phi^3) \wedge \omega^1 \\
& + f^2 (b\, dt \wedge \phi^2 - ac\phi^3 \wedge \phi^1) \wedge \omega^2 + f^2 (c\, dt \wedge \phi^3 - ab\phi^1 \wedge \phi^2) \wedge \omega^3.
\end{aligned}
\tag{3.98}
$$

A straightforward computation gives

$$
d\Psi = \Psi_0 dt \wedge \mathrm{vol}_B + \Psi_1 dt \wedge \phi^2 \wedge \phi^3 \wedge \omega^1 + \Psi_2 dt \wedge \phi^3 \wedge \phi^1 \wedge \omega^2 + \Psi_3 dt \wedge \phi^1 \wedge \phi^2 \wedge \omega^3 \,,
\tag{3.99}
$$

where

$$
\begin{aligned}
\Psi_0 &= -4f^3 \frac{df}{dt} - 2f^2(a + b + c), \\
\Psi_1 &= -abc + f^2 a - f^2 b - f^2 c - 2bcf\frac{df}{dt} - f^2 \frac{d(bc)}{dt} \,, \\
\Psi_2 &= -abc - f^2 a + f^2 b - f^2 c - 2acf\frac{df}{dt} - f^2 \frac{d(ac)}{dt} \,, \\
\Psi_3 &= -abc - f^2 a - f^2 b + f^2 c - 2abf\frac{df}{dt} - f^2 \frac{d(ab)}{dt} \,.
\end{aligned}
\tag{3.100}
$$

The Ricci-flat metrics with $Spin(7)$ holonomy satisfy the condition $d\Psi = 0$. In our case this is explicitly given by the following first-order differential equations [16]:

$$
\begin{aligned}
\frac{da}{dt} &= -\frac{a^2 - (b-c)^2}{2bc} + \frac{a^2}{f^2}\,, \\
\frac{db}{dt} &= -\frac{b^2 - (c-a)^2}{2ac} + \frac{b^2}{f^2}\,, \\
\frac{dc}{dt} &= -\frac{c^2 - (a-b)^2}{2ab} + \frac{c^2}{f^2}\,, \\
\frac{df}{dt} &= -\frac{a+b+c}{2f}\,.
\end{aligned}
\tag{3.101}
$$

Now we turn to the $Spin(7)$ with torsion equations (3.85)–(3.88). The dilaton Φ is calculated as

$$
\frac{d\Phi}{dt} = -\frac{1}{6f^2}\left(\frac{\Psi_0}{f^2} + \frac{2\Psi_1}{bc} + \frac{2\Psi_2}{ac} + \frac{2\Psi_3}{ab}\right) \tag{3.102}
$$

and the 3-form flux is given by

$$
H = H_\phi \phi^1 \wedge \phi^2 \wedge \phi^3 + \sum_{i=1}^{3} H_i \phi^i \wedge \omega^i\,, \tag{3.103}
$$

where

$$
\begin{aligned}
H_\phi &= -abc\left(\frac{\Psi_0}{f^4} + \frac{d\Phi}{dt}\right),\ H_1 = -af^2\left(\frac{\Psi_1}{bcf^2} + \frac{d\Phi}{dt}\right), \\
H_2 &= -bf^2\left(\frac{\Psi_2}{acf^2} + \frac{d\Phi}{dt}\right),\ H_3 = -cf^2\left(\frac{\Psi_3}{abf^2} + \frac{d\Phi}{dt}\right).
\end{aligned}
\tag{3.104}
$$

If the field strength F takes the form,

$$
F = \sum_{i=1}^{3} F_{ti} dt \wedge d\phi^i + \sum_{i=1}^{3} F_i \omega^i + \frac{1}{2}\sum_{i,j=1}^{3} F_{ij} \phi^i \wedge \phi^j\,, \tag{3.105}
$$

then the generalized self-dual equation (3.87) yields

$$
\frac{2F_1}{f^2} = -\frac{F_{23}}{bc} + \frac{F_{t1}}{a}\,,\ \frac{2F_2}{f^2} = -\frac{F_{31}}{ac} + \frac{F_{t2}}{b}\,,\ \frac{2F_3}{f^2} = -\frac{F_{12}}{ab} + \frac{F_{t3}}{c}. \tag{3.106}
$$

From (4.26) we obtain that

$$
\begin{aligned}
dH \;=\; & \frac{dH_\phi}{dt} dt \wedge \phi^1 \wedge \phi^2 \wedge \phi^3 + \sum_{i=1}^{3} \frac{dH_i}{dt} dt \wedge \phi^i \wedge \omega^i + 2\sum_{i=1}^{3} H_i\, vol_B \\
& + (H_\phi - H_1 + H_2 + H_3)\phi^2 \wedge \phi^3 \wedge \omega^1 + (H_\phi + H_1 - H_2 + H_3)\phi^3 \wedge \phi^1 \wedge \omega^2 \\
& + (H_\phi + H_1 + H_2 - H_3)\phi^1 \wedge \phi^2 \wedge \omega^3\,,
\end{aligned}
\tag{3.107}
$$

Using (3.88) and (3.102) we find that the solutions are classified into three types:

$$
\begin{aligned}
\text{(a)}\quad & F = d(F_1\phi^1)\,, \quad H = F_1\phi^1 \wedge F\,, \\
& \frac{dF_1}{dt} = aF_1\left(\frac{2}{f^2} - \frac{1}{bc}\right), \quad \frac{d\Phi}{dt} = -\frac{{F_1}^2}{a}\left(\frac{2}{f^2} - \frac{1}{bc}\right). \qquad (3.108)\\
\text{(b)}\quad & F = d(F_2\phi^2)\,, \quad H = F_2\phi^2 \wedge F\,, \\
& \frac{dF_2}{dt} = bF_2\left(\frac{2}{f^2} - \frac{1}{ac}\right), \quad \frac{d\Phi}{dt} = -\frac{{F_2}^2}{b}\left(\frac{2}{f^2} - \frac{1}{ac}\right). \qquad (3.109)\\
\text{(c)}\quad & F = d(F_3\phi^3)\,, \quad H = F_3\phi^3 \wedge F\,, \\
& \frac{dF_3}{dt} = cF_3\left(\frac{2}{f^2} - \frac{1}{ab}\right), \quad \frac{d\Phi}{dt} = -\frac{{F_3}^2}{c}\left(\frac{2}{f^2} - \frac{1}{ab}\right). \qquad (3.110)
\end{aligned}
$$

Then, for case (a), the $Spin(7)$ with torsion equation (3.85) reduces to the following differential equations:

$$
\begin{aligned}
\frac{da}{dt} &= -\frac{a^2 - (b-c)^2 - {F_1}^2}{2bc} + \frac{a^2 - {F_1}^2}{f^2}\,, \\
\frac{db}{dt} &= -\frac{b^2 - (c-a)^2 - {F_1}^2}{2ac} + \frac{b^2}{f^2}\,, \\
\frac{dc}{dt} &= -\frac{c^2 - (a-b)^2 - {F_1}^2}{2ab} + \frac{c^2}{f^2}\,, \\
\frac{df}{dt} &= -\frac{a+b+c}{2f} - \frac{F_1^2}{2af}\,, \qquad (3.111)
\end{aligned}
$$

and (b) and (c) are given by cyclic permutations of a, b, c.

Finally, we briefly discuss the solutions to (3.101) and the $Spin(7)$ with torsion equations given by (3.111). A more detail about the solutions will be reported elsewhere. The regular condition to the solutions requires the following boundary conditions at $t = 0$:

$$
\begin{aligned}
\text{(I)}\quad & a(0) = b(0) = c(0) = 0, \quad f(0) \neq 0, \\
& |a'(0)| = |b'(0)| = |c'(0)| = \frac{1}{2}, \quad f'(0) = 0, \qquad (3.112)\\
\text{(II)}\quad & a(0) = 0, \quad b(0) = -c(0), \quad f(0) \neq 0, \\
& |a'(0)| = 2, \quad b'(0) = c'(0), \quad f'(0) = 0. \qquad (3.113)
\end{aligned}
$$

The equation (3.101) gives Ricci-flat $Spin(7)$ holonomy metrics. Some explicit solutions satisfying the boundary conditions (I)(II) are constructed in [16, 17, 19, 20] with the help of numerical calculations, and further the existence of the regular solutions was analytically proved [39]. The $Spin(7)$ with torsion case (3.111) is slightly different from the Ricci-flat case. We find that case (II) admits no solution with non-zero F_1, while case (I) admits the following series solution around

$t = 0$:

$$a(t) = -\frac{t}{2} + a_3 t^3 + a_5 t^5 + \cdots ,$$
$$b(t) = -\frac{t}{2} + b_3 t^3 + b_5 t^5 + \cdots ,$$
$$c(t) = -\frac{t}{2} + c_3 t^3 + c_5 t^5 + \cdots ,$$
$$f(t) = f_0 + \frac{3}{8f_0} t^2 + f_4 t^4 + \cdots ,$$
$$F_1(t) = h_2 t^2 + h_4 t^4 + \cdots . \tag{3.114}$$

Here, the series have four independent parameters, $\{a_3, b_3, c_3, f_0, h_2\}$ with one constraint $a_3 + b_3 + c_3 = 2h_2^2 + 1/(4f_0^2)$, and higher coefficients are determined by these parameters. The series (3.114) can be extended to an ALC solution of (3.108) and (3.111) when the parameters are restricted to $b_3 = c_3$, and hence $b(t) = c(t)$ is satisfied for all t. Specifically, the explicit ALC solution with two parameters ℓ, k is given by

$$a(r) = -\frac{\ell(r-\ell)\sqrt{(r-3\ell)(r+\ell)}}{(1+k^2\ell^2)(r-\ell)^2 - 4k^2\ell^4},$$
$$b(r) = c(r) = -\frac{1}{2}\sqrt{(r-3\ell)(r+\ell)},$$
$$f(r) = \sqrt{\frac{r^2-\ell^2}{2}} \tag{3.115}$$

together with

$$F_1(r) = \frac{k\ell^2(r-3\ell)(r+\ell)}{(1+k^2\ell^2)(r-\ell)^2 - 4k^2\ell^4}. \tag{3.116}$$

Here, we used a radial coordinate r $(r \geq 3\ell)$ defined by $dt = (r-\ell)dr/\sqrt{(r-3\ell)(r+\ell)}$.

3.4 Conclusion

We have derived G_2T equations in Abelian heterotic supergravity theory. When a G_2 manifold is locally given by $\boldsymbol{R}_+ \times S^3 \times S^3$, the G_2T equations are reduced to ordinary differential equations(3.42)–(3.49). A numerical analysis of these equations shows that a global solution doesn't exist for the general six radial functions $a_i(t), b_i(t)$ $(i = 1, \cdots, 3)$ and thus we study the reduced case, $a_1(t) = a_2(t)$ and $b_1(t) = b_2(t)$. The Abelian heterotic solutions (g, H, φ, F) are obtained from (3.70)–(3.73) by using Ricci-flat G_2 holonomy metrics. To construct regular solutions of the reduced G_2T equations, we have investigated S^3-bolt solutions and $T^{1,1}$-bolt solutions by numerical analysis. These solutions are shown graphically in figures 1 and 2. The formulas (3.70)–(3.73) generate only S^3-bolt solutions. A problem of finding analytic expressions for $T^{1,1}$-bolt solutions remains as a future work. We have also derived $Spin(7)$ with torsion equations based on 3-Sasakian manifolds. The explicit ALC solution to Abelian heterotic supergravity theory has been obtained from these equations.

Chapter 4

Supersymmetric heterotic solutions with $SU(3)$-structure

This chapter is based on Ref. [1]. We consider a geometry of a NS sector in $E_8 \times E_8$ heterotic string theory. This theory has Bianchi identity,

$$dH = \alpha'(tr\mathcal{F} \wedge \mathcal{F} - tr\mathcal{R}^- \wedge \mathcal{R}^-), \tag{4.1}$$

and this constraints the back ground geometry. Here $\mathcal{R}^-$ is a curvature of Hull connection ∇^-, which is given by

$$\nabla^- = \nabla - \frac{1}{2}H\,, \tag{4.2}$$

where ∇ is a Levi-Civitá connection. On the other hand, Bismut connection ∇^+ appears in the Killing spinor equations arising from supersymmetry variations of the gravitino and it is given by

$$\nabla^+ = \nabla + \frac{1}{2}H\,. \tag{4.3}$$

We assume that ten-dimensional space-time takes the form $\boldsymbol{R}^{1,3} \times M_6$, where M_6 is a six-dimensional space with Killing spinors obeying the Strominger system. Thus we should consider a geometry on M_6. If a field strength is identified with a curvature of a Hull connection, $\mathcal{F} = \mathcal{R}^-$, a part of the gauge symmetry $E_8 \times E_8$ is embedded into the background geometry, which is known as the standard embedding especially in the case $H = 0$ [42]. In $H = 0$ case, the holonomy of $\nabla^\pm$ is in $SU(3)$ and thus a part of the gauge symmetry is $SU(3)$ by the standard embedding. Therefore $E_8 \times E_8$ breaks to $E_6 \times E_8$ and this is known as a hallmark of Calabi–Yau compactification. In $H \neq 0$ case, the holonomy of ∇^+ belongs to $SU(3)$, whereas the holonomy of ∇^- is in $SO(6)$ generally. Hence $E_8 \times E_8$ may break to $SO(10) \times E_8$ by the standard embedding and this is desirable in a phenomenological view.

In this chapter, we construct a supersymmetric heterotic supergravity solution such that a holonomy of Bismut connection ∇^+ is in $SU(3)$ but a holonomy of Hull connection ∇^- is *not*, by superposing two hyper-Kählers with torsion (HKT) geometries. As already pointed out in Ref. [40], one can obtain HKT geometries by conformally transforming hyper-Kähler geometries. We choose the Gibbons–Hawking space as the starting point and apply a conformal transformation to obtain a HKT geometry. Then, we superpose these two HKT geometries and require a resulting metric for some conditions. Consequently, we find that the holonomy of Hull connection on the superposed solution remains to be $SO(4)$. We also show that by T dualizing this solution the holonomy of Hull connection turns into $SO(5)$ or $SO(6)$.

We take a two-dimensional periodic array of the "intersecting HKT" solutions to get a compact six-dimensional solution. We find that the fundamental parallelogram of the two-dimensional periodic array is separated into distinct smooth regions bordered by codimension-1 singularity hypersurfaces, hence the name "supersymmetric domain wall."

4.1 HKT geometry as a conformal transform

We start with a four-dimensional hyper-Kähler with torsion (HKT) metric g_{HKT} obtained as a conformal transform of a hyper-Kähler metric, where for the latter we specifically consider the Gibbons–Hawking (GH) metric g_{GH},

$$g_{HKT} = \Phi \, g_{GH}. \tag{4.4}$$

The GH metric is given by [41]

$$g_{GH} = \frac{1}{\phi}\left(d\tau - \sum_{i=1}^{3} \psi_i dx^i\right)^2 + \phi \sum_{i=1}^{3} (dx^i)^2 \,, \tag{4.5}$$

where ϕ and $\boldsymbol{\psi} = (\psi_1, \psi_2, \psi_3)$ are scalar functions of the coordinates (x^1, x^2, x^3) of $\mathbf{R}^3$ obeying the relation

$$\mathrm{grad}\, \phi = \mathrm{rot}\, \boldsymbol{\psi}. \tag{4.6}$$

Φ is a scalar field of which the properties will be described shortly. We define the orthonormal basis

$$E^0 = \sqrt{\frac{\Phi}{\phi}}\left(d\tau - \sum_{i=1}^{3} \psi_i dx^i\right), \;\; E^i = \sqrt{\Phi\, \phi}\, dx^i \;\; (i = 1, 2, 3), \tag{4.7}$$

so that the hypercomplex structure is given by the three complex structures J^a $(a = 1, 2, 3)$ satisfying the quaternionic identities,

$$J^a(E^\mu) = \bar{\eta}^a_{\mu\nu} E^\nu, \tag{4.8}$$

where $\bar{\eta}^a_{\mu\nu}$ are the 't Hooft matrices. The corresponding fundamental 2-forms are

$$\Omega^a = -\bar{\eta}^a_{\mu\nu} E^\mu \wedge E^\nu. \tag{4.9}$$

The HKT structure is defined by the 3-form torsion T satisfying [43, 44]

$$T = J^1 d\Omega^1 = J^2 d\Omega^2 = J^3 d\Omega^3 \,. \tag{4.10}$$

In the present case, we have

$$T = -E_0 \log \Phi E^{123} + E_1 \log \Phi E^{023} + E_2 \log \Phi E^{031} + E_3 \log \Phi E^{012} \tag{4.11}$$

in terms of dual vector fields E_μ to the 1-forms (4.7),

$$E_0 = \sqrt{\frac{\phi}{\Phi}} \frac{\partial}{\partial \tau}, \quad E_i = \frac{1}{\sqrt{\Phi\,\phi}} \left(\frac{\partial}{\partial x^i} + \psi_i \frac{\partial}{\partial \tau} \right) \tag{4.12}$$

and

$$E^{\mu\nu\lambda} = E^\mu \wedge E^\nu \wedge E^\lambda. \tag{4.13}$$

The exterior derivative is calculated as

$$dT = -\frac{1}{\Phi^2 \phi} \left(\sum_{\mu=0}^{3} V_\mu^2 \Phi \right) E^{0123} \tag{4.14}$$

with the vector fields $V_\mu = \sqrt{\Phi\,\phi}\, E_\mu$. Therefore, if Φ is chosen to be a harmonic function with respect to the GH metric (4.5), then the torsion T becomes a closed 3-form.

Using this T, we introduce the two types of connections $\nabla^\pm$,

$$\nabla^\pm_X Y = \nabla_X Y \pm \frac{1}{2} \sum_{\mu=0}^{3} T(X, Y, E_\mu) E_\mu, \tag{4.15}$$

where ∇ is a Levi-Civitá connection. The corresponding connection 1-forms $\omega^{\pm\,\mu}{}_{\nu}$ are defined by

$$\nabla^\pm_{E_\mu} E_\nu = \omega^{\pm\,\lambda}{}_{\nu}(E_\mu) E_\lambda, \tag{4.16}$$

and the curvature 2-forms are written as

$$\mathcal{R}^{\pm\,\mu}{}_{\nu} = d\omega^{\pm\,\mu}{}_{\nu} + \omega^{\pm\,\mu}{}_{\lambda} \wedge \omega^{\pm\,\lambda}{}_{\nu}. \tag{4.17}$$

The torsion curvature $\mathcal{R}^{+\,\mu}{}_{\nu}$ satisfies the $SU(2)$ holonomy condition

$$\mathcal{R}^{+}{}_{01} + \mathcal{R}^{+}{}_{23} = 0,\ \mathcal{R}^{+}{}_{02} + \mathcal{R}^{+}{}_{31} = 0,\ \mathcal{R}^{+}{}_{03} + \mathcal{R}^{+}{}_{12} = 0. \tag{4.18}$$

On the other hand, if the torsion T is a closed 3-form, that is, Φ is a harmonic function, then the curvature $\mathcal{R}^{-\,\mu}{}_{\nu}$ becomes an Anti–Self–Dual 2-form, which may be regarded as a Yang–Mills instanton with the gauge group $SU(2) \times SU(2) = SO(4)$.

4.2 Intersecting HKT metrics

In the previous section we have seen that the HKT metrics obtained by a conformal transformation have $\omega^{+\mu}{}_{\nu}$ in $SU(2)$ but $\omega^{-\mu}{}_{\nu}$ in $SO(4)$ strictly larger than $SU(2)$ as long as the original GH space is not a flat Euclidean space. In this section we construct their six-dimensional analogs by superposing two such HKT metrics embedded in different four-dimensional subspaces. This construction is motivated by that used in constructing intersecting brane solutions [45, 46] [1]; namely, we assume the form of the metric as

$$g = \Phi\tilde{\Phi}\phi\tilde{\phi}((dx^1)^2+(dx^2)^2)+\Phi\phi(dx^3)^2+\frac{\Phi}{\phi}(dx^4-\psi dx^3)^2+\tilde{\Phi}\tilde{\phi}(dx^5)^2+\frac{\tilde{\Phi}}{\tilde{\phi}}(dx^6-\tilde{\psi}dx^5)^2\,. \quad (4.19)$$

The HKT metric that we have considered in the previous section is characterized by a triplet (Φ, ϕ, ψ) on $\mathbf{R}^3 = \{(x^1, x^2, x^3)\}$ obeying (4.6). So at first it might seem that (Φ, ϕ) or $(\tilde{\Phi}, \tilde{\phi})$ could to be functions of (x^1, x^2, x^3) or (x^1, x^2, x^5), and $dx^4 - \psi dx^3$ or $dx^6 - \tilde{\psi}dx^5$ could be replaced with a more general form $dx^4 - \sum_{i=1,2,3}\psi_i dx^i$ or $dx^6 - \sum_{i=1,2,5}\tilde{\psi}_i dx^i$, respectively. However, it turns out that such a more general ansatz does not lead to a metric with $SU(3)$ holonomy even in the case $\Phi = \tilde{\Phi} = 1$. Thus we are led to consider the metric of the form (4.19), assuming the following:

- (Φ, ϕ) and $(\tilde{\Phi}, \tilde{\phi})$ are harmonic functions on the two-dimensional flat space $\mathbf{R^2} = \{(x^1, x^2)\}$.
- $\boldsymbol{\psi} = (0, 0, \psi)$ and $\tilde{\boldsymbol{\psi}} = (0, 0, \tilde{\psi})$, of which the components are harmonic functions on $\mathbf{R^2}$ satisfying the Cauchy–Riemann conditions

$$\begin{aligned} \frac{\partial\phi}{\partial x_2} &= -\frac{\partial\psi}{\partial x_1}, \quad \frac{\partial\phi}{\partial x_1} = \frac{\partial\psi}{\partial x_2}, \\ \frac{\partial\tilde{\phi}}{\partial x_2} &= -\frac{\partial\tilde{\psi}}{\partial x_1}, \quad \frac{\partial\tilde{\phi}}{\partial x_1} = \frac{\partial\tilde{\psi}}{\partial x_2}. \end{aligned} \quad (4.20)$$

Under these assumptions, we will show that a six-dimensional space M_6 with the metric (4.19) has the following KT structure:

(a) a closed Bismut torsion [see, Eq. (4.26)],

(b) an exact Lee form [see, Eq. (4.27)],

(c) a Bismut connection ∇^+ with $SU(3)$ holonomy [see, Eq. (4.34)].

We first introduce an orthonormal basis

$$\begin{aligned} &e^1 = \sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}dx^1\ ,\ e^2 = \sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}dx^2\,,\ e^3 = \sqrt{\Phi\phi}dx^3\ ,\ e^4 = \sqrt{\frac{\Phi}{\phi}}(dx^4-\psi dx^3)\,, \\ &e^5 = \sqrt{\tilde{\Phi}\tilde{\phi}}dx^5\ ,\ e^6 = \sqrt{\frac{\tilde{\Phi}}{\tilde{\phi}}}(dx^6-\tilde{\psi}dx^5) \end{aligned} \quad (4.21)$$

[1] The term “intersecting” in the (commonly used) name is misleading since they are smeared and hence do not have intersections with larger codimensions. See, e.g., Ref. [47] for recent developments in constructing localized intersecting brane solutions in supergravity.

and their exterior derivatives de^a are as follows:

$$
\begin{aligned}
de^1 &= -\frac{1}{\Phi\tilde{\Phi}\phi\tilde{\phi}}\partial_2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}e^{12}\,,\ de^2 = \frac{1}{\Phi\tilde{\Phi}\phi\tilde{\phi}}\partial_1\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}e^{12}\,,\\
de^3 &= \frac{1}{2\Phi\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}[(\phi\partial_1\Phi+\Phi\partial_1\phi)e^{13}+(\phi\partial_2\Phi+\Phi\partial_2\phi)e^{23}]\,,\\
de^4 &= \frac{1}{2\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left\{\left(\partial_1\Phi-\frac{\Phi}{\phi}\partial_1\phi\right)e^{14}+\left(\partial_2\Phi-\frac{\Phi}{\phi}\partial_2\phi\right)e^{24}\right\}-\frac{1}{\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\psi e^{13}+\partial_2\psi e^{23})\,,\\
de^5 &= \frac{1}{2\tilde{\Phi}\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}[(\tilde{\phi}\partial_1\tilde{\Phi}+\tilde{\Phi}\partial_1\tilde{\phi})e^{15}+(\tilde{\phi}\partial_2\tilde{\Phi}+\tilde{\Phi}\partial_2\tilde{\phi})e^{25}]\,,\\
de^6 &= \frac{1}{2\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left\{\left(\partial_1\tilde{\Phi}-\frac{\tilde{\Phi}}{\tilde{\phi}}\partial_1\tilde{\phi}\right)e^{16}+\left(\partial_2\tilde{\Phi}-\frac{\tilde{\Phi}}{\tilde{\phi}}\partial_2\tilde{\phi}\right)e^{26}\right\}-\frac{1}{\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\tilde{\psi}e^{15}+\partial_2\tilde{\psi}e^{25})\,.
\end{aligned}
\tag{4.22}
$$

The space M_6 has a natural complex structure J defined by

$$J(e^1)=\epsilon_1 e^2\,,\ J(e^3)=\epsilon_2 e^4\,,\ J(e^5)=\epsilon_3 e^6 \tag{4.23}$$

together with the conditions $|\epsilon_i|=1(i=1,2,3)$ and $\epsilon_1\epsilon_2=\epsilon_1\epsilon_3=-1$. Indeed, we can directly check that a Nijenhuis tensor associated with J vanishes under these conditions (see Appendix A). Then, the metric (4.19) becomes Hermitian with respect to the complex structure J, and the related fundamental 2-form κ takes the form

$$\kappa=\epsilon_1 e^1\wedge e^2+\epsilon_2 e^3\wedge e^4+\epsilon_3 e^5\wedge e^6\,. \tag{4.24}$$

The Bismut torsion T is uniquely determined by

$$\nabla^+_X g=0\,,\ \nabla^+_X\kappa=0\,. \tag{4.25}$$

Explicitly, we have

$$
\begin{aligned}
T &= -J\,d\kappa\\
&= \frac{1}{\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\Phi e^{234}-\partial_2\Phi e^{134})+\frac{1}{\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\tilde{\Phi}e^{256}-\partial_2\tilde{\Phi}e^{156}).
\end{aligned}
\tag{4.26}
$$

It should be noticed that in our case the Bismut torsion is a closed 3-form, $dT=0$. We shall refer to ∇^+ and ∇^- as the Bismut connection and Hull connection, respectively, according to Ref. [10]. The Lee form θ is a 1-form defined by $\Theta=-J\delta\kappa$ [11], which becomes a exact 1-form,

$$\Theta=2d\varphi\,,\ \varphi=\log\sqrt{\Phi\tilde{\Phi}}. \tag{4.27}$$

The space M_6 admits a $SU(3)$-structure, which is classified by

$$d(e^{-2\varphi}\Upsilon) = 0\,, \tag{4.28}$$
$$d(e^{-2\varphi} * \kappa) = 0\,, \tag{4.29}$$
$$T = -e^{2\varphi} * d(e^{-2\varphi}\kappa)\,, \tag{4.30}$$

where Υ is the complex $(3,0)$-form associated with κ and it is explicitly given by

$$\Upsilon = \sqrt{\epsilon_1}(e^1 + i\,e^2) \wedge \sqrt{\epsilon_2}(e^3 - i\,e^4) \wedge \sqrt{\epsilon_3}(e^5 - i\,e^6)\,. \tag{4.31}$$

The triplet (κ, Υ, T) satisfies these equations (see Appendix C). We will identify the Bismut torsion with 3-form flux, $T = H$, and the function φ with a dilaton. The equations (4.28), (4.29), and (4.30) are defining equations of the $SU(3)$-structure admitted by a Calabi–Yau with torsion manifold.

We should study the holonomy of the Bismut connection. The spin connection with the Bismut torsion $\omega^+_{\mu\nu}$ are as follows:

$$\omega^+_{12} = \frac{1}{2\Phi\tilde{\Phi}\phi\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\big(\,\partial_2(\Phi\tilde{\Phi}\phi\tilde{\phi})e^1 - \partial_1(\Phi\tilde{\Phi}\phi\tilde{\phi})e^2\,\big)\,,$$

$$\omega^+_{13} = -\frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\frac{1}{\Phi\phi}\partial_1(\Phi\phi)e^3 + \left(\frac{1}{\phi}\partial_2\log\phi - \frac{1}{\Phi}\partial_2\log\Phi\right)e^4\right)\,,$$

$$\omega^+_{14} = -\frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\left(\frac{1}{\phi}\partial_2\phi + \frac{1}{\Phi}\partial_2\Phi\right)e^3 + \left(\frac{1}{\Phi}\partial_1\Phi - \frac{1}{\phi}\partial_1\phi\right)e^4\right)\,,$$

$$\omega^+_{15} = -\frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\frac{1}{\tilde{\Phi}\tilde{\phi}}\partial_1(\tilde{\Phi}\tilde{\phi})e^5 + \left(\frac{1}{\tilde{\phi}}\partial_2\tilde{\phi} - \frac{1}{\tilde{\Phi}}\partial_2\tilde{\Phi}\right)e^6\right)\,,$$

$$\omega^+_{16} = -\frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\left(\frac{1}{\tilde{\phi}}\partial_2\tilde{\phi} + \frac{1}{\tilde{\Phi}}\partial_2\tilde{\Phi}\right)e^5 + \left(\frac{1}{\tilde{\Phi}}\partial_1\tilde{\Phi} - \frac{1}{\tilde{\phi}}\partial_1\tilde{\phi}\right)e^6\right)\,,$$

$$\omega^+_{23} = \frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(-\frac{1}{\Phi\phi}\partial_2(\Phi\phi)e^3 + \left(\frac{1}{\phi}\partial_1\phi - \frac{1}{\Phi}\partial_1\Phi\right)e^4\right)\,,$$

$$\omega^+_{24} = \frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\left(\frac{1}{\phi}\partial_1\phi + \frac{1}{\Phi}\partial_1\Phi\right)e^3 - \left(\frac{1}{\Phi}\partial_2\Phi - \frac{1}{\phi}\partial_2\phi\right)e^4\right)\,,$$

$$\omega^+_{25} = \frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(-\frac{1}{\tilde{\Phi}\tilde{\phi}}\partial_2(\tilde{\Phi}\tilde{\phi})e^5 + \left(\frac{1}{\tilde{\phi}}\partial_1\tilde{\phi} - \frac{1}{\tilde{\Phi}}\partial_1\tilde{\Phi}\right)e^6\right)\,,$$

$$\omega_{26}^{+} = \frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\left(\frac{1}{\tilde{\phi}}\partial_1\tilde{\phi} + \frac{1}{\tilde{\Phi}}\partial_1\tilde{\Phi}\right)e^5 - \left(\frac{1}{\tilde{\Phi}}\partial_2\tilde{\Phi} - \frac{1}{\tilde{\phi}}\partial_2\tilde{\phi}\right)e^6\right),$$

$$\omega_{34}^{+} = \frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\left(\frac{1}{\phi}\partial_2\phi + \frac{1}{\Phi}\partial_2\Phi\right)e^1 - \left(\frac{1}{\phi}\partial_1\phi + \frac{1}{\Phi}\partial_1\Phi\right)e^2\right),$$

$$\omega_{35}^{+} = 0\,,\ \omega_{36}^{+} = 0\,,\ \omega_{45}^{+} = 0\,,\ \omega_{46}^{+} = 0\,,$$

$$\omega_{56}^{+} = \frac{1}{2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\left(\frac{1}{\tilde{\phi}}\partial_2\tilde{\phi} + \frac{1}{\tilde{\Phi}}\partial_2\tilde{\Phi}\right)e^1 - \left(\frac{1}{\tilde{\phi}}\partial_1\tilde{\phi} + \frac{1}{\tilde{\Phi}}\partial_1\tilde{\Phi}\right)e^2\right). \tag{4.32}$$

By the definition of spin connection with torsion, $\omega_{\mu\nu}^{+}$ for each μ, ν are clearly skew symmetric for exchanging μ with ν and thus non-trivial $\omega_{\mu\nu}^{+}$ are above all. The spin connection 1-forms $\omega_{\mu\nu}^{+}$ clearly satisfy the condition

$$\sum_{\mu,\nu=1}^{6} \omega_{\mu\nu}^{+}\kappa_{\mu\nu} = 2(\epsilon_1\omega_{12}^{+} + \epsilon_2\omega_{34}^{+} + \epsilon_3\omega_{56}^{+}) = 0\,, \tag{4.33}$$

which is equivalent to the condition (see Sec. 2.1)

$$\epsilon_1\mathcal{R}_{12}^{+} + \epsilon_2\mathcal{R}_{34}^{+} + \epsilon_3\mathcal{R}_{56}^{+} = 0\,. \tag{4.34}$$

Therefore the Ricci form of the Bismut connection vanishes and this is implies that the holonomy of ∇^{+} is contained in $SU(3)$ [11]. In addition, M_6 admits two independent Killing spinors obeying $\nabla_X^{+}\varepsilon = 0$ in type II theory.Thus, the triplet (g, H, φ) gives rise to a supersymmetric solution to the type II supergravity theory.

4.3 Embedding into heterotic string theory and T-duality

We study supersymmetric solutions describing heterotic flux compactification. The bosonic part of the string frame action, up to the first order in the α' expansion, is given by

$$S = \frac{1}{2\kappa^2}\int d^{10}x\sqrt{-g}e^{-2\varphi}\left(R + 4(\nabla\varphi)^2 - \frac{1}{12}H_{MNP}H^{MNP} - \alpha'(tr\mathcal{F}_{MN}\mathcal{F}^{MN} - tr\mathcal{R}_{MN}^{-}\mathcal{R}^{-MN})\right). \tag{4.35}$$

It is assumed that ten-dimensional spacetimes take the form $R^{1,3} \times M_6$, where M_6 is a six-dimensional space admitting a Killing spinor ε,

$$\nabla_\mu^{+}\varepsilon = 0, \quad \left(\gamma^\mu\partial_\mu\varphi - \frac{1}{12}H_{\mu\nu\rho}\gamma^{\mu\nu\rho}\right)\varepsilon = 0, \quad \mathcal{F}_{\mu\nu}\gamma^{\mu\nu}\varepsilon = 0. \tag{4.36}$$

This system together with the anomaly cancellation condition

$$dH = \alpha'\left(tr\mathcal{F}\wedge\mathcal{F} - tr\mathcal{R}^{-}\wedge\mathcal{R}^{-}\right) \tag{4.37}$$

is known as the Strominger system [48].

Now, we turn to the heterotic solution obeying the Strominger system. If the curvature $\mathcal{R}^-$ in the anomaly condition (4.37) is given by the Hull connection ∇^-, we can choose a non-Abelian gauge field as $\mathcal{F} = \mathcal{R}^-$ since the 3-form flux (4.26) is closed by the identification $T = H$. This is a form of the usual standard embedding. Combining the well-known identity

$$\mathcal{R}^+_{ab\mu\nu} - \mathcal{R}^-_{\mu\nu ab} = \frac{1}{2}(dT)_{ab\mu\nu} = 0 \tag{4.38}$$

with the holonomy condition (4.34), we can see that the gauge field $\mathcal{F}$ is an instanton satisfying the third equation in (4.36).

Apparently, $\mathcal{F}$ seems to take values in $SO(6) \subset E_8$, which would describe a symmetry breaking from E_8 to $SO(10)$. However, for generic choices of the harmonic functions ϕ, Φ, $\tilde{\phi}$ and $\tilde{\Phi}$, it is not ensured that the metric (4.19) can remain non-negative, and the dilaton (4.27) can remain real valued. Therefore, to get a meaningful solution, we are forced to impose

$$\phi = \tilde{\phi} = \Phi = \tilde{\Phi}\,. \tag{4.39}$$

Under the condition (4.39), we study the holonomy of the Bismut connection and the Hull connection. In case of the Bismut connection, $\epsilon_4 = -1$, these connections are given by

$$\begin{aligned}
\omega^+_{12} &= \frac{2}{\phi^3}(\,\partial_2\phi e^1 - \partial_1\phi e^2\,)\,,\; \omega^+_{34} = \frac{1}{\phi^3}(\,\partial_2\phi e^1 - \partial_1\phi e^2\,)\,, \\
\omega^+_{56} &= \frac{1}{\phi^3}(\,\partial_2\phi e^1 - \partial_1\phi e^2\,)\,,\; \omega^+_{35} = 0\,,\; \omega^+_{36} = 0\,,\; \omega^+_{45} = 0\,,\; \omega^+_{46} = 0\,, \\
\omega^+_{13} &= -\frac{1}{\phi^3}\partial_1\phi e^3\,,\; \omega^+_{14} = -\frac{1}{\phi^3}\partial_2\phi e^3\,,\; \omega^+_{15} = -\frac{1}{\phi^3}\partial_1\phi e^5\,, \\
\omega^+_{16} &= -\frac{1}{\phi^3}\partial_2\phi e^5\,,\; \omega^+_{23} = -\frac{1}{\phi^3}\partial_2\phi e^3\,,\; \omega^+_{24} = \frac{1}{\phi^3}\partial_1\phi e^3\,, \\
\omega^+_{25} &= -\frac{1}{\phi^3}\partial_2\phi e^5\,,\; \omega^+_{26} = \frac{1}{\phi^3}\partial_1\phi e^5
\end{aligned} \tag{4.40}$$

and these teach us the relations

$$\omega^+_{13} = -\omega^+_{24}\,,\; \omega^+_{14} = \omega^+_{23}\,,\; \omega^+_{15} = -\omega^+_{26}\,,\; \omega^+_{16} = \omega^+_{25}\,,\; \omega^+_{12} = \omega^+_{34} + \omega^+_{56}\,. \tag{4.41}$$

The relation $\omega^+_{12} = \omega^+_{34} + \omega^+_{56}$ give a relation $\mathcal{R}^+_{12} = \mathcal{R}^+_{34} + \mathcal{R}^+_{56}$ and thus the Ricci form of the Bismut connection vanishes. The Bismut curvature 2-form $\mathcal{R}^+ = d\omega^+ + \omega^+ \wedge \omega^+$ is

$$\begin{aligned}
\mathcal{R}^+ &= \frac{1}{2}\mathcal{R}^+_{ab}M^{ab} \\
&= \mathcal{R}^+_{13}(M^{13} - M^{24}) + \mathcal{R}^+_{14}(M^{14} + M^{23}) + \mathcal{R}^+_{15}(M^{15} - M^{26}) + \mathcal{R}^+_{16}(M^{16} + M^{25}) \\
&\quad + \mathcal{R}^+_{35}(M^{35} + M^{46}) + \mathcal{R}^+_{36}(M^{36} - M^{45})
\end{aligned} \tag{4.42}$$

and commutation relations of their generators are following:

$$
\begin{aligned}
&[M^{12}+M^{34}\,,\,M^{12}+M^{56}]=0\,,\ [M^{12}+M^{34}\,,\,M^{13}-M^{24}]=-2(M^{14}+M^{23})\,,\\
&[M^{12}+M^{34}\,,\,M^{14}+M^{23}]=2(M^{13}+M^{24})\,,\ [M^{12}+M^{34}\,,\,M^{15}-M^{26}]=-(M^{16}+M^{25})\,,\\
&[M^{12}+M^{34}\,,\,M^{16}+M^{25}]=(M^{15}-M^{26})\,,\ [M^{12}+M^{56}\,,\,M^{13}-M^{24}]=-(M^{14}+M^{23})\,,\\
&[M^{12}+M^{56}\,,\,M^{14}+M^{23}]=(M^{13}-M^{24})\,,\ [M^{12}+M^{56}\,,\,M^{15}-M^{26}]=-2(M^{16}+M^{25})\,,\\
&[M^{12}+M^{56}\,,\,M^{16}+M^{25}]=2(M^{15}-M^{26})\,,\ [M^{13}-M^{24}\,,\,M^{14}+M^{23}]=-(M^{12}+M^{34})\\
&[M^{13}-M^{24}\,,\,M^{15}-M^{26}]=-(M^{35}+M^{46})\,,\ [M^{13}+M^{24}\,,\,M^{16}+M^{25}]=-(M^{36}-M^{45})\\
&[M^{14}+M^{23}\,,\,M^{15}-M^{26}]=(M^{36}-M^{45})\,,\ [M^{14}+M^{23}\,,\,M^{16}+M^{25}]=-(M^{35}+M^{46})\\
&[M^{15}-M^{26}\,,\,M^{16}+M^{25}]=-2(M^{12}+M^{56})\,,
\end{aligned}
\tag{4.43}
$$

where $\mathfrak{so}(6)$ generators M^{ab} $(a,b=1,\cdots,6)$ satisfy the commutation relation

$$
[M^{ab},M^{cd}]=\delta^{bc}M^{ad}-\delta^{ac}M^{bd}-\delta^{bd}M^{ac}+\delta^{ad}M^{bc}\,. \tag{4.44}
$$

Thus eight independent generators $M^{12}+M^{34}$, $M^{12}+M^{56}$, $M^{13}-M^{24}$, $M^{14}+M^{23}$, $M^{15}-M^{26}$, $M^{16}+M^{25}$, $M^{35}+M^{46}$, and $M^{36}-M^{45}$ generate $\mathfrak{su}(3)$ Lie algebra. Therefore the holonomy of the Bismut connection is contained in $SU(3)$. On the other hand, the case of the Hull connection, $\epsilon_4=1$, these connections are given by

$$
\begin{aligned}
&\omega^-_{12}=\frac{2}{\phi^3}(\,\partial_2\phi e^1-\partial_1\phi e^2\,)\,,\ \omega^-_{13}=-\frac{1}{\phi^3}(\,\partial_1\phi e^3+\partial_2\phi e^4\,)\,,\\
&\omega^-_{15}=-\frac{1}{\phi^3}(\,\partial_1\phi e^5+\partial_2\phi e^6\,)\,,\ \omega^-_{23}=\frac{1}{\phi^3}(\,-\partial_2\phi e^3+\partial_1\phi e^4\,)\,,\\
&\omega^-_{25}=\frac{1}{\phi^3}(\,-\partial_2\phi e^5+\partial_1\phi e^6\,)\,,\\
&\omega^-_{14}=\omega^-_{16}=\omega^-_{24}=\omega^-_{26}=\omega^-_{34}=\omega^-_{35}=\omega^-_{36}=\omega^-_{45}=\omega^-_{46}=\omega^-_{56}=0\,.
\end{aligned}
\tag{4.45}
$$

The Hull curvature 2-form $\mathcal{R}^-=d\omega^-+\omega^-\wedge\omega^-$ is

$$
\mathcal{R}^-=\frac{1}{2}\mathcal{R}^-_{ab}M^{ab}=\mathcal{R}^-_{12}M^{12}+\mathcal{R}^-_{13}M^{13}+\mathcal{R}^-_{15}M^{15}+\mathcal{R}^-_{23}M^{23}+\mathcal{R}^-_{25}M^{25}+\mathcal{R}^-_{35}M^{35} \tag{4.46}
$$

and commutation relations of their generators are the following:

$$
\begin{aligned}
&[M^{12}\,,\,M^{13}]=-M^{23}\ ,\ [M^{12}\,,\,M^{15}]=-M^{25}\ ,\ [M^{12}\,,\,M^{23}]=M^{13}\ ,\ [M^{12}\,,\,M^{25}]=M^{15}\\
&[M^{13}\,,\,M^{15}]=-M^{35}\ ,\ [M^{13}\,,\,M^{23}]=-M^{12}\ ,\ [M^{13}\,,\,M^{25}]=0\ ,\ [M^{15}\,,\,M^{23}]=0\\
&[M^{23}\,,\,M^{25}]=-M^{35}\ ,\ [M^{35}\,,\,M^{13}]=-M^{15}\ ,\ [M^{35}\,,\,M^{15}]=M^{13}\ ,\ [M^{35}\,,\,M^{23}]=-M^{25}\\
&[M^{35}\,,\,M^{25}]=-M^{23}\,.
\end{aligned}
$$

Thus six independent generators M_{12}, M_{13}, M_{15}, M_{23}, M_{25}, and M_{35} generate the Lie algebra $\mathfrak{so}(4)$.Consequently, with the condition (4.39), the holonomy of ∇^+ remains $SU(3)$, whereas the instanton $\mathcal{F}$ reduces to a proper Lie subalgebra $SO(4)$ of $SO(6)$.

To recover the $SO(6)$ instanton, we apply a T-duality transformation. From (4.19), (4.26), and (4.27) with $\phi = \tilde{\phi} = \Phi = \tilde{\Phi}$, we have the following metric with the holonomy of ∇^+ which is contained in $SU(3)$, 3-form flux, and dilaton:

$$
\begin{aligned}
g &= \phi^4((dx^1)^2 + (dx^2)^2) + \phi^2(\,(dx^3)^2 + (dx^5)^2\,) + (dx^4 - \psi dx^3)^2 + (dx^6 - \psi dx^5)^2\,, && (4.47)\\
H &= -\frac{1}{\phi^3}(\partial_2\phi e^{134} - \partial_1\phi e^{234} + \partial_2\phi e^{156} - \partial_1\phi e^{256})\,, && (4.48)\\
\varphi &= \frac{1}{2}\log\phi^2\,. && (4.49)
\end{aligned}
$$

The metric (4.47) has isometries $U(1)^4$ generated by Killing vector fields ∂_a $(a = 3,4,5,6)$. Therefore, we can T-dualize the type II solution (g, H, φ) along directions of these isometries. It is easy to see that the solution is inert under the T-duality along x^4 and x^6; the T-dualities along the remaining directions give nontrivial deformations of the solutions, preserving one-quarter of supersymmetries. [2]

We first T-dualize the solution along x^3. The resulting solution $(\hat{g}, \hat{H}, \hat{\varphi})$ is given by

$$
\begin{aligned}
\hat{g} &= \phi^4(\,(dx^1)^2 + (dx^2)^2\,) + \frac{1}{\phi^2+\psi^2}(d\hat{x}^3 + \psi dx^4)^2 \\
&\quad + \frac{\phi^2}{\phi^2+\psi^2}(dx^4)^2 + \phi^2(dx^5)^2 + (dx^6 - \psi dx^5)^2\,, \qquad (4.50)
\end{aligned}
$$

$$
\begin{aligned}
\hat{H} &= \frac{1}{\phi^3(\phi^2+\psi^2)}(\,(\,(\phi^2-\psi^2)\partial_2\phi + 2\phi\psi\partial_1\phi\,)\hat{e}^{134} - (\,(\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi\,)\hat{e}^{234}\,) \\
&\quad -\frac{1}{\phi^3}(\partial_2\phi\hat{e}^{156} - \partial_1\phi\hat{e}^{256})\,, \qquad (4.51)
\end{aligned}
$$

$$
\hat{\varphi} = \frac{1}{2}\log\left(\frac{\phi^2}{\phi^2+\psi^2}\right)\,. \qquad (4.52)
$$

Here, the orthonormal basis is defined by

$$
\begin{aligned}
&\hat{e}^1 = \phi^2 dx^1\,,\ \hat{e}^2 = \phi^2 dx^2\,,\ \hat{e}^3 = \frac{1}{\sqrt{\phi^2+\psi^2}}\left(d\hat{x}^3 + \psi dx^4\right)\,,\ \hat{e}^4 = \frac{\phi}{\sqrt{\phi^2+\psi^2}}dx^4\,,\\
&\hat{e}^5 = \phi dx^5\,,\ \hat{e}^6 = dx^6 - \psi dx^5 \qquad (4.53)
\end{aligned}
$$

and their exterior derivatives are given by

$$
\begin{aligned}
d\hat{e}^1 &= -2\phi\frac{\partial\phi}{\partial x^2}dx^{12} = -\frac{2}{\phi^3}\frac{\partial\phi}{\partial x^2}\hat{e}^{12}\,,\ d\hat{e}^2 = \frac{2}{\phi^3}\frac{\partial\phi}{\partial x^1}\hat{e}^{12}\,,\\
d\hat{e}^3 &= \frac{1}{\phi^2(\phi^2+\psi^2)}(\,-(\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^{13} - (\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^{23}\,) - \frac{1}{\phi^3}\partial_2\phi\hat{e}^{14} + \frac{1}{\phi^3}\partial_1\phi\hat{e}^{24}\,,\\
d\hat{e}^4 &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\psi\partial_1\phi + \phi\partial_2\phi)\hat{e}^{14} + (\psi\partial_2\phi - \phi\partial_1\phi)\hat{e}^{24}\,)\,,\\
d\hat{e}^5 &= \frac{1}{\phi^3}(\partial_1\phi\hat{e}^{15} + \partial_2\phi\hat{e}^{25})\,,\ d\hat{e}^6 = \frac{1}{\phi^3}(\partial_2\phi\hat{e}^{15} - \partial_1\phi\hat{e}^{25})\,. \qquad (4.54)
\end{aligned}
$$

[2] See, e.g., Ref. [49] for the classification of supersymmetric solutions to heterotic supergravity.

Then, we have a deformed complex structure $\hat{J}$,

$$\hat{J}\hat{e}_1 = \epsilon_1 \hat{e}_2 \ , \ \hat{J}\hat{e}_3 = \epsilon_2 \hat{e}_4 \ , \ \hat{J}\hat{e}_5 = \epsilon_3 \hat{e}_6 \tag{4.55}$$

with $|\epsilon_i| = 1 (i = 1, 2, 3)$ and $\epsilon_1\epsilon_2 = \epsilon_1\epsilon_3 = -1$. Indeed, a Nijenhuis tensor related with $\hat{J}$ vanishes under the conditions (see Appendix D). The associated fundamental 2-form $\hat{\kappa}$ takes the same form as (4.24),

$$\hat{\kappa} = \epsilon_1 \hat{e}^1 \wedge \hat{e}^2 + \epsilon_2 \hat{e}^3 \wedge \hat{e}^4 + \epsilon_3 \hat{e}^5 \wedge \hat{e}^6 \tag{4.56}$$

and the associated complex $(3, 0)$-form $\hat{\Upsilon}$ is given by

$$\hat{\Upsilon} = i \frac{\phi + i\psi}{\sqrt{\phi^2 + \psi^2}} \left(\hat{e}^{135} + \hat{e}^{245} + \hat{e}^{236} - \hat{e}^{146} + i(\hat{e}^{235} - \hat{e}^{145} - \hat{e}^{136} - \hat{e}^{246}) \right) . \tag{4.57}$$

A Bismut torsion $\hat{T}$ is defined by $\hat{T} = -\hat{J}_* d\hat{\kappa}$. Then, we find that the torsion $\hat{T}$ is a closed 3-form and $\hat{T} = \hat{H}$. An Lee form $\hat{\Theta}$ is defined by $\hat{\Theta} = -\hat{J}_* \delta\hat{\kappa}$ and it takes the form, $\hat{\Theta} = 2d\hat{\varphi}$, which is clearly an exact 1-form. A $SU(3)$ structure admitted on the space M_6 is defined by the same form equations as (4.28), (4.29), and (4.30) and $(\hat{\kappa}, \hat{T}, \hat{\Upsilon})$ satisfies these equations (see Appendix D). This suggests that the space $(M_6, \hat{g}, \hat{\kappa}, \hat{\Upsilon})$ is a Calabi–Yau with torsion manifold. We should confirm that the Ricci form of the Bismut connection $\hat{\nabla}^+$ vanishes. The Bismut connection 1-forms $\hat{\omega}^+_{\mu\nu}$ are presented as follows:

$$\begin{aligned}
&\hat{\omega}^+_{12} = \omega^+_{12} \, , \, \hat{\omega}^+_{56} = \omega^+_{56} \, , \, \hat{\omega}^+_{34} = \frac{\psi}{2\phi} d \log{(\phi^2 + \psi^2)} \, , \\
&\hat{\omega}^+_{35} = 0 \ , \ \hat{\omega}^+_{36} = 0 \ , \ \hat{\omega}^+_{45} = 0 \ , \ \hat{\omega}^+_{46} = 0 \, , \\
&\hat{\omega}^+_{13} = \frac{1}{\phi^2(\phi^2 + \psi^2)} \left((\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^3 + \frac{\psi}{\phi}(\psi\partial_2\phi - \phi\partial_1\phi)\hat{e}^4 \right) \, , \\
&\hat{\omega}^+_{14} = \frac{1}{\phi^2(\phi^2 + \psi^2)} \left((\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^3 - \frac{\psi}{\phi}(\psi\partial_1\phi + \phi\partial_2\phi)\hat{e}^4 \right) \, , \\
&\hat{\omega}^+_{23} = \frac{1}{\phi^2(\phi^2 + \psi^2)} \left((\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^3 - \frac{\psi}{\phi}(\psi\partial_1\phi + \phi\partial_2\phi)\hat{e}^4 \right) \, , \\
&\hat{\omega}^+_{24} = -\frac{1}{\phi^2(\phi^2 + \psi^2)} \left((\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^3 + \frac{\psi}{\phi}(\psi\partial_2\phi - \phi\partial_1\phi)\hat{e}^4 \right) \, , \\
&\hat{\omega}^+_{15} = -\frac{1}{\phi^3}\partial_1\phi\hat{e}^5 \, , \, \hat{\omega}^+_{16} = -\frac{1}{\phi^3}\partial_2\phi\hat{e}^5 \, , \, \hat{\omega}^+_{25} = -\frac{1}{\phi^3}\partial_2\phi\hat{e}^5 \ , \ \hat{\omega}^+_{26} = \frac{1}{\phi^3}\partial_1\phi\hat{e}^5 \, .
\end{aligned} \tag{4.58}$$

We notice that following four relations hold as same as the Bismut connection 1-forms for before the T-duality transformation along ∂_3,

$$\hat{\omega}^+_{13} = -\hat{\omega}^+_{24} \ , \ \hat{\omega}^+_{14} = \hat{\omega}^+_{23} \ , \ \hat{\omega}^+_{15} = -\hat{\omega}^+_{26} \ , \ \hat{\omega}^+_{16} = \hat{\omega}^+_{25} \, , \tag{4.59}$$

but $\hat{\omega}^+_{12} \neq \hat{\omega}^+_{34} + \hat{\omega}^+_{56}$. Here we can rewrite $\hat{\omega}^+_{34}$ as follows,

$$\begin{aligned}
\hat{\omega}^+_{34} &= \frac{\psi}{\phi(\phi^2+\psi^2)} \left(\frac{\psi}{\phi} \partial_2 \log\phi - \partial_1 \log\phi \right) \hat{e}^1 - \frac{\psi}{\phi(\phi^2+\psi^2)} \left(\frac{\psi}{\phi} \partial_1 \log\phi + \partial_2 \log\phi \right) \hat{e}^2 \\
&= \frac{1}{\phi^2} (\partial_2 \log\phi \hat{e}^1 - \partial_1 \log\phi \hat{e}^2) \\
&\quad - \frac{1}{\phi(\phi^2+\psi^2)} ((\phi\partial_2 \log\phi + \psi\partial_1 \log\phi)\hat{e}^1 - (\phi\partial_1 \log\phi - \psi\partial_2 \log\phi)\hat{e}^2) \\
&= \omega^+_{34} - \Lambda \,,
\end{aligned} \tag{4.60}$$

where the 1-form Λ is defined by

$$\Lambda = \frac{1}{\phi^2(\phi^2+\psi^2)} ((\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^1 - (\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^2) \,. \tag{4.61}$$

Then, we obtain

$$\sum_{\mu<\nu} \hat{\omega}^+_{\mu\nu} \hat{\kappa}_{\mu\nu} = \epsilon_1 \hat{\omega}^+_{12} + \epsilon_2 \hat{\omega}^+_{34} + \epsilon_3 \hat{\omega}^+_{56} = -\omega^+_{12} + \omega^+_{34} - \Lambda + \omega^+_{56} = -\Lambda \tag{4.62}$$

Therefore we don't see whether the Ricci form of the Bismut connection $\hat{\nabla}^+$ vanishes or not in the view of the connection. We need to investigate it at the level of the Bismut curvature. The condition $\hat{\rho}(X,Y) = 0$ is equivalent to $\sum_{\mu,\nu} \hat{\mathcal{R}}^+_{\mu\nu} \hat{\kappa}_{\mu\nu} = 0$, where $\hat{\mathcal{R}}^+$ is curvature 2-form related with the Bismut connection $\hat{\omega}^+$. $\sum_{\mu,\nu} \hat{\mathcal{R}}^+_{\mu\nu} \hat{\kappa}_{\mu\nu}$ is given by

$$\sum_{\mu,\nu=1}^{6} \hat{\mathcal{R}}^+_{\mu\nu} \hat{\kappa}_{\mu\nu} = \epsilon_1 \hat{\mathcal{R}}^+_{12} + \epsilon_2 \hat{\mathcal{R}}^+_{34} + \epsilon_3 \hat{\mathcal{R}}^+_{56} = -\hat{\mathcal{R}}^+_{12} + \hat{\mathcal{R}}^+_{34} + \hat{\mathcal{R}}^+_{56} \,, \tag{4.63}$$

where each terms are

$$\begin{aligned}
\hat{\mathcal{R}}^+_{12} &= d\hat{\omega}^+_{12} + \sum_\rho \hat{\omega}^+_{1\rho} \wedge \hat{\omega}^+_{\rho 2} \;=\; -\frac{2}{\phi^4} \left[\{ \partial_1^2 (\log\phi) + \partial_2^2 (\log\phi) \right] \hat{e}^{12} \;=\; \mathcal{R}^+_{12} \\
\hat{\mathcal{R}}^+_{34} &= d\omega^+_{34} - d\Lambda \;=\; \mathcal{R}^+_{34} - d\Lambda \;=\; -\frac{1}{\phi^4} (\partial_1^2 \log\phi + \partial_2^2 \log\phi) \hat{e}^{12} - d\Lambda \,, \\
\hat{\mathcal{R}}^+_{56} &= -\frac{1}{\phi^4} (\partial_1^2 \log\phi + \partial_2^2 \log\phi) \hat{e}^{12} \;=\; \mathcal{R}^+_{56} \,,
\end{aligned} \tag{4.64}$$

We see that the 1-from Λ is closed-form as follows,

$$\begin{aligned}
d\Lambda &= -\frac{2}{\phi^4(\phi^2+\psi^2)} \{ (\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^1 + (\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^2 \} \wedge \{ (\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^1 - (\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^2 \} \\
&\quad - \frac{1}{\phi^6(\phi^2+\psi^2)} \{ \partial_1(\phi^2)\hat{e}^1 + \partial_2(\phi^2)\hat{e}^2 \} \wedge \{ (\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^1 - (\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^2 \} \\
&\quad - \frac{2}{\phi^4(\phi^2+\psi^2)} \{ (\partial_2\phi)^2 + (\partial_1\phi)^2 \} \hat{e}^{12} \\
&\quad - \frac{2}{\phi^6(\phi^2+\psi^2)} (\, (\phi\partial_2\phi + \psi\partial_1\phi)(\partial_2\phi)\hat{e}^{12} + (\phi\partial_1\phi - \psi\partial_2\phi)(\partial_1\phi)\hat{e}^{12} \,) \\
&= 0 \,.
\end{aligned}$$

Hence we have

$$\sum_{\mu,\nu=1}^{6} \hat{\mathcal{R}}^{+}_{\mu\nu}\hat{\kappa}_{\mu\nu} = -\hat{\mathcal{R}}^{+}_{12} + \hat{\mathcal{R}}^{+}_{34} + \hat{\mathcal{R}}^{+}_{56} = 0\,. \tag{4.65}$$

Thus the Ricci form of the Bismut connection $\hat{\nabla}^{+}$ vanishes. The Bismut curvature 2-form $\hat{\mathcal{R}}^{+}$ takes the same form as (4.42).Therefore the Bismut connection $\hat{\nabla}^{+}$ has an $SU(3)$ holonomy.

We consider the holonomy of the Hull connection $\hat{\nabla}^{-}$. The Hull connection 1-forms $\hat{\omega}^{-}_{\mu\nu}$ are presented as follows:

$$\begin{aligned}
&\hat{\omega}^{-}_{12} = \omega^{-}_{12}\,,\ \hat{\omega}^{-}_{34} = \frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^1 - (\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^2\,)\,,\\
&\hat{\omega}^{-}_{13} = \frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^3 + (\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^4\,)\,,\\
&\hat{\omega}^{-}_{14} = -\frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^3 + (\psi\partial_1\phi + \phi\partial_2\phi)\hat{e}^4\,)\,,\\
&\hat{\omega}^{-}_{23} = \frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^3 - (\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^4\,)\,,\\
&\hat{\omega}^{-}_{24} = \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^3 - (\psi\partial_2\phi - \phi\partial_1\phi)\hat{e}^4\,)\,,\\
&\hat{\omega}^{-}_{15} = -\frac{1}{\phi^3}(\partial_1\phi\hat{e}^5 + \partial_2\phi\hat{e}^6)\,,\ \hat{\omega}^{-}_{25} = -\frac{1}{\phi^3}(\partial_2\phi\hat{e}^5 - \partial_1\phi\hat{e}^6)\,,\\
&\hat{\omega}^{-}_{16} = \hat{\omega}^{-}_{26} = \hat{\omega}^{-}_{35} = \hat{\omega}^{-}_{36} = \hat{\omega}^{-}_{45} = \hat{\omega}^{-}_{46} = \hat{\omega}^{-}_{56} = 0\,.
\end{aligned} \tag{4.66}$$

We introduce $\mathfrak{so}(6)$ generators $M^{ab}\,(a,b=1,\cdots,6)$ satisfying (4.44). Then, the curvature 2-form $\hat{\mathcal{R}}^{-}$ (see Appendix D) is given by

$$\hat{\mathcal{R}}^{-} = \hat{\mathcal{R}}^{-}_{12}M^{12} + \hat{\mathcal{R}}^{-}_{13}M^{13} + \hat{\mathcal{R}}^{-}_{14}M^{14} + \hat{\mathcal{R}}^{-}_{15}M^{15} + \hat{\mathcal{R}}^{-}_{23}M^{23} + \hat{\mathcal{R}}^{-}_{24}M^{24} + \hat{\mathcal{R}}^{-}_{25}M^{25} + \hat{\mathcal{R}}^{-}_{35}M^{35} + \hat{\mathcal{R}}^{-}_{45}M^{45} \tag{4.67}$$

and commutation relations to the generators M^{ab} are the following:

$$\begin{aligned}
&[M^{12}\,,\,M^{13}] = -M^{23}\,,\ [M^{12}\,,\,M^{14}] = -M^{24}\,,\ [M^{12}\,,\,M^{15}] = -M^{25}\,,\ [M^{12}\,,\,M^{23}] = M^{13}\\
&[M^{12}\,,\,M^{24}] = M^{14}\,,\ [M^{12}\,,\,M^{25}] = M^{15}\,,\ [M^{12}\,,\,M^{34}] = 0\,,\ [M^{12}\,,\,M^{35}] = 0\\
&[M^{12}\,,\,M^{45}] = 0\,,\ [M^{13}\,,\,M^{14}] = -M^{34}\ ,\ [M^{13}\,,\,M^{15}] = -M^{35}\,,\ [M^{13}\,,\,M^{23}] = -M^{12}\\
&[M^{13}\,,\,M^{24}] = 0\,,\ [M^{13}\,,\,M^{25}] = 0\ [M^{13}\,,\,M^{34}] = M^{14}\ ,\ [M^{13}\,,\,M^{35}] = M^{15}\\
&[M^{13}\,,\,M^{45}] = 0\,,\ [M^{14}\,,\,M^{15}] = -M^{45}\,,\ [M^{14}\,,\,M^{23}] = 0\,,\ [M^{14}\,,\,M^{24}] = -M^{12}\\
&[M^{14}\,,\,M^{25}] = 0\,,\ [M^{14}\,,\,M^{34}] = -M^{13}\ ,\ [M^{14}\,,\,M^{35}] = 0\ ,\ [M^{14}\,,\,M^{45}] = M^{15}\\
&[M^{15}\,,\,M^{23}] = 0\,,\ [M^{15}\,,\,M^{24}] = 0\,,\ [M^{15}\,,\,M^{34}] = 0\,,\ [M^{15}\,,\,M^{35}] = -M^{13}\\
&[M^{15}\,,\,M^{45}] = -M^{14}\,,\ [M^{23}\,,\,M^{24}] = -M^{34}\,,\ [M^{23}\,,\,M^{25}] = -M^{35}\,,\ [M^{23}\,,\,M^{34}] = M^{24}\\
&[M^{23}\,,\,M^{35}] = M^{25}\,,\ [M^{23}\,,\,M^{45}] = 0\,,\ [M^{24}\,,\,M^{25}] = -M^{45}\,,\ [M^{24}\,,\,M^{34}] = -M^{23}\\
&[M^{24}\,,\,M^{35}] = 0\,,\ [M^{24}\,,\,M^{45}] = M^{25}\,,\ [M^{34}\,,\,M^{35}] = -M^{45}\ ,\ [M^{34}\,,\,M^{45}] = M^{35}\\
&[M^{35}\,,\,M^{45}] = -M^{34}\,.
\end{aligned} \tag{4.68}$$

These ten independent generators generate the Lie algebra $\mathfrak{so}(5)$. In this case, it turns out that the Hull connection $\hat{\nabla}^-$ is in $SO(5)$, which is still smaller than $SO(6)$.

Thus, we further T-dualize the solution $(\hat{g}, \hat{H}, \hat{\varphi})$ once more along x^5 and finally obtain $(\tilde{g}, \tilde{H}, \tilde{\varphi})$:

$$\begin{aligned} \tilde{g} &= \phi^4(\,(dx^1)^2 + (dx^2)^2\,) + \frac{1}{\phi^2+\psi^2}(d\hat{x}^3 + \psi dx^4)^2 \\ &\quad + \frac{1}{\phi^2+\psi^2}(d\tilde{x}^5 + \psi dx^6)^2 + \frac{\phi^2}{\phi^2+\psi^2}(\,(dx^4)^2 + (dx^6)^2\,)\,, \end{aligned} \tag{4.69}$$

$$\begin{aligned} \tilde{H} &= \frac{1}{\phi^3(\phi^2+\psi^2)}(\,((\phi^2-\psi^2)\partial_2\phi + 2\phi\psi\partial_1\phi)\tilde{e}^{134} - ((\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi)\tilde{e}^{234}\,) \\ &\quad + \frac{1}{\phi^3(\phi^2+\psi^2)}(\,((\phi^2-\psi^2)\partial_2\phi + 2\phi\psi\partial_1\phi)\tilde{e}^{156} - ((\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi)\tilde{e}^{256}\,), \end{aligned} \tag{4.70}$$

$$\tilde{\varphi} = \frac{1}{2}\log\left(\frac{\phi^2}{(\phi^2+\psi^2)^2}\right)\,. \tag{4.71}$$

The orthonormal basis is defined by

$$\begin{aligned} &\tilde{e}^1 = \hat{e}^1\,,\ \tilde{e}^2 = \hat{e}^2\,,\ \tilde{e}^3 = \hat{e}^3\,,\ \tilde{e}^4 = \hat{e}^4\,, \\ &\tilde{e}^5 = \frac{1}{\sqrt{\phi^2+\psi^2}}(d\tilde{x}^5 + \psi dx^6)\,,\ \tilde{e}^6 = \frac{\phi}{\sqrt{\phi^2+\psi^2}}dx^6 \end{aligned} \tag{4.72}$$

and exterior derivatives of them are as follows:

$$\begin{aligned} d\tilde{e}^1 &= -\frac{2}{\phi^3}\frac{\partial\phi}{\partial x^2}\tilde{e}^{12}\,,\ d\tilde{e}^2 = \frac{2}{\phi^3}\frac{\partial\phi}{\partial x^1}\tilde{e}^{12}\,, \\ d\tilde{e}^3 &= -\frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^{13} - (\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^{23}\,) - \frac{1}{\phi^3}\partial_2\phi\tilde{e}^{14} + \frac{1}{\phi^3}\partial_1\phi\tilde{e}^{24}\,, \\ d\tilde{e}^4 &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\psi\partial_1\phi + \phi\partial_2\phi)\tilde{e}^{14} + (\psi\partial_2\phi - \phi\partial_1\phi)\tilde{e}^{24}\,)\,, \\ d\tilde{e}^5 &= \frac{1}{\phi^2(\phi^2+\psi^2)}(\,-(\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^{15} - (\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^{25}\,) - \frac{1}{\phi^3}\partial_1 B_2\tilde{e}^{16} - \frac{1}{\phi^3}\partial_2 B_2\tilde{e}^{26}\,, \\ d\tilde{e}^6 &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\psi\partial_1\phi + \phi\partial_2\phi)\tilde{e}^{16} + (\psi\partial_2\phi - \phi\partial_1\phi)\tilde{e}^{26}\,)\,. \end{aligned} \tag{4.73}$$

In this basis the complex structure $\tilde{J}$ is given by

$$\tilde{J}\tilde{e}^1 = \epsilon_1\tilde{e}^2\ ,\ \tilde{J}\tilde{e}^3 = \epsilon_2\tilde{e}^4\ ,\ \tilde{J}\tilde{e}^4 = \epsilon_3\tilde{e}^2 \tag{4.74}$$

with $|\epsilon_i| = 1 (i = 1,2,3)$ and $\epsilon_1\epsilon_2 = \epsilon_1\epsilon_3 = -1$. Indeed, a Nijenhuis tensor associated with $\tilde{J}$ vanishes under the conditions (see Appendix E). The fundamental 2-form $\tilde{\kappa}$ takes the form

$$\tilde{\kappa} = \epsilon_1\tilde{e}^1\wedge\tilde{e}^2 + \epsilon_2\tilde{e}^3\wedge\tilde{e}^4 + \epsilon_3\tilde{e}^5\wedge\tilde{e}^6 \tag{4.75}$$

and the associated complex $(3,0)$-form $\tilde{\Upsilon}$ is given by

$$\tilde{\Upsilon} = i\frac{\phi + i\psi}{\phi^2 + \psi^2}(\,\hat{e}^{135} + \tilde{e}^{245} + \tilde{e}^{236} - \tilde{e}^{146} + i(\tilde{e}^{235} - \tilde{e}^{145} - \tilde{e}^{136} - \tilde{e}^{246})\,)\,. \tag{4.76}$$

A Bismut torsion $\tilde{T}$ is defined by $\tilde{T} = -\tilde{J}_* d\tilde{\kappa}$, which is a closed 3-form and satisfies $\tilde{T} = \tilde{H}$. An Lee form $\tilde{\Theta}$ is defined by $\tilde{\Theta} = -\tilde{J}_*\delta\tilde{\kappa}$ and it takes the form, $\tilde{\Theta} = 2d\tilde{\varphi}$. A $SU(3)$ structure admitted on the space M_6 is classified by the same form equations as (4.28), (4.29), and (4.30). Then, $(\tilde{\kappa}, \tilde{T}, \tilde{\Upsilon})$ satisfies these equations (see Appendix E). This implies that the space $(M_6, \tilde{g}, \tilde{\kappa}, \tilde{\Upsilon})$ is a Calabi–Yau with torsion manifold. We should verify that the Ricci form of the Bismut connection $\tilde{\nabla}^+$ vanishes. The Bismut connection 1-forms $\tilde{\omega}^+_{\mu\nu}$ are presented as follows:

$$\tilde{\omega}^+_{12} = \omega^+_{12}\,,\ \tilde{\omega}^+_{34} = \hat{\omega}^+_{34}\,,\ \tilde{\omega}^+_{56} = \frac{\psi}{2\phi}d\log{(\phi^2 + \psi^2)}$$

$$\tilde{\omega}^+_{13} = \hat{\omega}^+_{13}\,,\ \tilde{\omega}^+_{14} = \hat{\omega}^+_{14}\,,\ \tilde{\omega}^+_{23} = \hat{\omega}^+_{23}\,,\ \tilde{\omega}^+_{24} = \hat{\omega}^+_{24}\,,$$

$$\begin{aligned}
\tilde{\omega}^+_{15} &= \frac{1}{\phi^2(\phi^2+\psi^2)}\left((\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^5 + \frac{\psi}{\phi}(\psi\partial_2\phi - \phi\partial_1\phi)\tilde{e}^6\right) \\
\tilde{\omega}^+_{16} &= \frac{1}{\phi^2(\phi^2+\psi^2)}\left((\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^5 - \frac{\psi}{\phi}(\psi\partial_1\phi + \phi\partial_2\phi)\tilde{e}^6\right) \\
\tilde{\omega}^+_{25} &= \frac{1}{\phi^2(\phi^2+\psi^2)}\left((\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^5 - \frac{\psi}{\phi}(\psi\partial_1\phi + \phi\partial_2\phi)\tilde{e}^6\right) \\
\tilde{\omega}^+_{26} &= -\frac{1}{\phi^2(\phi^2+\psi^2)}\left((\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^5 + \frac{\psi}{\phi}(\psi\partial_2\phi - \phi\partial_1\phi)\tilde{e}^6\right)
\end{aligned}$$

$$\tilde{\omega}^+_{35} = 0\ ,\ \tilde{\omega}^+_{36} = 0\ ,\ \tilde{\omega}^+_{45} = 0\ ,\ \tilde{\omega}^+_{46} = 0$$

and we notice that following four relations hold as same as the Bismut connection for before the T-duality transformation,

$$\tilde{\omega}^+_{13} = -\tilde{\omega}^+_{24}\,,\ \tilde{\omega}^+_{14} = \tilde{\omega}^+_{23}\,,\ \tilde{\omega}^+_{15} = -\tilde{\omega}^+_{26}\,,\ \tilde{\omega}^+_{16} = \tilde{\omega}^+_{25}\,, \tag{4.77}$$

but $\tilde{\omega}^+_{12} \neq \tilde{\omega}^+_{34} + \tilde{\omega}^+_{56}$. As it was already mentioned, $\tilde{\omega}^+_{34}$ is given by $\tilde{\omega}^+_{34} = \omega^+_{34} - \Lambda$, where Λ is defined by (4.61). Also, $\tilde{\omega}^+_{56}$ is rewritten as follows,

$$\begin{aligned}
\tilde{\omega}^+_{56} &= \frac{1}{\phi^2}(\partial_2\log\phi\tilde{e}^1 - \partial_1\log\phi\tilde{e}^2) \\
&\quad - \frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^1 - (\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^2\,) \\
&= \omega^+_{56} - \Lambda\,.
\end{aligned} \tag{4.78}$$

The relation $\tilde{\omega}^+_{12} = \tilde{\omega}^+_{34} + \tilde{\omega}^+_{56}$ is not valid,

$$\sum_{\mu<\nu} \tilde{\omega}^+_{\mu\nu}\tilde{\kappa}_{\mu\nu} = \epsilon_1\tilde{\omega}^+_{12} + \epsilon_2\tilde{\omega}^+_{34} + \epsilon_3\tilde{\omega}^+_{56} = -\omega^{T_B}_{12} + \omega^{T_B}_{34} - \Lambda + \omega^{T_B}_{56} - \Lambda = -2\Lambda\,.$$

Therefore we don't see whether the Ricci form $\tilde{\rho}$ of the Bismut connection $\tilde{\nabla}^+$ vanishes in the view of the connection. We need to investigate it at the level of the Bismut curvature. The condition $\tilde{\rho} = 0$ is equivalent to $\sum_{\mu,\nu} \tilde{\mathcal{R}}^+_{\mu\nu}\tilde{\kappa}_{\mu\nu} = 0$, where $\tilde{\mathcal{R}}^+$ is curvature 2-form related with the Bismut connection $\tilde{\omega}^+$. Hence we confirm that $\sum_{\mu,\nu} \tilde{\mathcal{R}}^+_{\mu\nu}\tilde{\kappa}_{\mu\nu}$ vanishes. $\sum_{\mu,\nu} \tilde{\mathcal{R}}^+_{\mu\nu}\tilde{\kappa}_{\mu\nu}$ is given by

$$\sum_{\mu,\nu=1}^{6} \tilde{\mathcal{R}}^+_{\mu\nu}\tilde{\kappa}_{\mu\nu} \;=\; \epsilon_1\tilde{\mathcal{R}}^+_{12} + \epsilon_2\tilde{\mathcal{R}}^+_{34} + \epsilon_3\tilde{\mathcal{R}}^+_{56} \;=\; -\tilde{\mathcal{R}}^+_{12} + \tilde{\mathcal{R}}^+_{34} + \tilde{\mathcal{R}}^+_{56}\,, \tag{4.79}$$

where each terms are following:

$$\begin{aligned}
\tilde{\mathcal{R}}^+_{12} &= -\frac{2}{\phi^4}(\,\partial_1^2\log\phi + \partial_2^2\log\phi\,)\tilde{e}^{12} \;=\; \mathcal{R}^+_{12}\,, \\
\tilde{\mathcal{R}}^+_{34} &= \mathcal{R}^+_{34} - d\Lambda \;=\; -\frac{1}{\phi^4}(\partial_1^2\log\phi + \partial_2^2\log\phi)\tilde{e}^{12} - d\Lambda\,, \\
\tilde{\mathcal{R}}^+_{56} &= d\omega^+_{56} - d\Lambda \;=\; \mathcal{R}^+_{56} - d\Lambda \;=\; -\frac{1}{\phi^4}(\partial_1^2\log\phi + \partial_2^2\log\phi)\tilde{e}^{12} - d\Lambda\,.
\end{aligned} \tag{4.80}$$

As was already mentioned, the 1-form Λ is closed-form. Thus we have

$$\sum_{\mu,\nu=1}^{6} \tilde{\mathcal{R}}^+_{\mu\nu}\tilde{\kappa}_{\mu\nu} = -\tilde{\mathcal{R}}^+_{12} + \tilde{\mathcal{R}}^+_{34} + \tilde{\mathcal{R}}^+_{56} = 0 \tag{4.81}$$

and thus the Ricci form of the Bismut connection $\tilde{\nabla}^+$ vanishes. The curvature 2-form $\tilde{\mathcal{R}}^+$ also takes the same form as (4.42). Therefore the Bismut holonomy is contained in $SU(3)$.

We should investigate the holonomy of the Hull connection $\tilde{\nabla}^-$. The Hull connection 1-forms $\tilde{\omega}^-_{\mu\nu}$ are presented as follows:

$$\begin{aligned}
&\tilde{\omega}^-_{12} = \hat{\omega}^-_{12}\,, \tilde{\omega}^-_{34} = \hat{\omega}^-_{34}\,, \tilde{\omega}^-_{35} = \tilde{\omega}^-_{36} = \tilde{\omega}^-_{45} = \tilde{\omega}^-_{46} = 0\,, \\
&\tilde{\omega}^-_{56} = \frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^1 - (\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^2\,)\,,
\end{aligned}$$

$$\begin{aligned}
&\tilde{\omega}^-_{13} = \hat{\omega}^-_{13}\,, \tilde{\omega}^-_{14} = \hat{\omega}^-_{14}\,, \tilde{\omega}^-_{23} = \hat{\omega}^-_{23}\,, \tilde{\omega}^-_{24} = \hat{\omega}^-_{24}\,, \\
&\tilde{\omega}^-_{15} = \frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^5 + (\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^6\,)\,, \\
&\tilde{\omega}^-_{16} = -\frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^5 + (\psi\partial_1\phi + \phi\partial_2\phi)\tilde{e}^6\,)\,, \\
&\tilde{\omega}^-_{25} = \frac{1}{\phi^2(\phi^2+\psi^2)}(\,(\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^5 - (\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^6\,)\,,
\end{aligned}$$

$$\tilde{\omega}^-_{26} = \frac{\psi}{\phi^3(\phi^2+\psi^2)}\big(\,(\phi\partial_2\phi+\psi\partial_1\phi)\tilde{e}^5-(\psi\partial_2\phi-\phi\partial_1\phi)\tilde{e}^6\,\big)\,. \tag{4.82}$$

We introduce $\mathfrak{so}(6)$ generators $M^{ab}\,(a,b=1,\cdots,6)$ satisfying (4.44). Then, the Hull curvature 2-form $\tilde{\mathcal{R}}^-$ (see Appendix E) is given by

$$\begin{aligned}\tilde{\mathcal{R}}^- \;=\;& \tilde{\mathcal{R}}^-_{12}M^{12}+\tilde{\mathcal{R}}^-_{13}M^{13}+\tilde{\mathcal{R}}^-_{14}M^{14}+\tilde{\mathcal{R}}^-_{15}M^{15}+\tilde{\mathcal{R}}^-_{16}M^{16}+\tilde{\mathcal{R}}^-_{23}M^{23}+\tilde{\mathcal{R}}^-_{24}M^{24}\\ &+\tilde{\mathcal{R}}^-_{25}M^{25}+\tilde{\mathcal{R}}^-_{26}M^{26}+\tilde{\mathcal{R}}^-_{35}M^{35}+\tilde{\omega}^-_{36}M^{36}+\tilde{\mathcal{R}}^-_{45}M^{45}+\tilde{\omega}^-_{46}M^{46}\end{aligned} \tag{4.83}$$

and commutation relations to the generators M^{ab} are the following:

$$\begin{aligned}
&[M^{12}\,,\,M^{13}]=-M^{23}\,,\,[M^{12}\,,\,M^{14}]=-M^{24}\,,\,[M^{12}\,,\,M^{15}]=-M^{25}\,,\,[M^{12}\,,\,M^{16}]=-M^{26}\\
&[M^{12}\,,\,M^{23}]=M^{13}\,,\,[M^{12}\,,\,M^{24}]=M^{14}\,,\,[M^{12}\,,\,M^{25}]=M^{15}\,,\,[M^{12}\,,\,M^{26}]=M^{16}\\
&[M^{12}\,,\,M^{34}]=0\,,\,[M^{12}\,,\,M^{35}]=0\,,\,[M^{12}\,,\,M^{45}]=0\,,\,[M^{12}\,,\,M^{46}]=0\,,\,[M^{12}\,,\,M^{56}]=0\\
&[M^{13}\,,\,M^{14}]=-M^{34}\,,\,[M^{13}\,,\,M^{15}]=-M^{35}\,,\,[M^{13}\,,\,M^{16}]=-M^{36}\;[M^{13}\,,\,M^{23}]=-M^{12}\\
&[M^{13}\,,\,M^{24}]=0\,,\,[M^{13}\,,\,M^{25}]=0\,,\,[M^{13}\,,\,M^{26}]=0\,,\,[M^{13}\,,\,M^{34}]=M^{14}\\
&[M^{13}\,,\,M^{35}]=M^{15}\,,\,[M^{13}\,,\,M^{36}]=M^{16}\,,\,[M^{13}\,,\,M^{45}]=0\,,\,[M^{13}\,,\,M^{45}]=0\\
&[M^{13}\,,\,M^{56}]=0\,,\,[M^{14}\,,\,M^{15}]=-M^{45}\,,\,[M^{14}\,,\,M^{16}]=-M^{46}\,,\,[M^{14}\,,\,M^{23}]=0\\
&[M^{14}\,,\,M^{24}]=-M^{12}\,,\,[M^{14}\,,\,M^{25}]=0\,,\,[M^{14}\,,\,M^{26}]=0\,,\,[M^{14}\,,\,M^{34}]=-M^{13}\\
&[M^{14}\,,\,M^{35}]=0\,,\,[M^{14}\,,\,M^{36}]=0\,,\,[M^{14}\,,\,M^{45}]=M^{15}\,,\,[M^{14}\,,\,M^{46}]=M^{16}\\
&[M^{14}\,,\,M^{56}]=0\,,\,[M^{15}\,,\,M^{16}]=-M^{56}\,,\,[M^{15}\,,\,M^{23}]=0\,,\,[M^{15}\,,\,M^{24}]=0\\
&[M^{15}\,,\,M^{25}]=-M^{12}\,,\,[M^{15}\,,\,M^{26}]=0\,,\,[M^{15}\,,\,M^{34}]=0\,,\,[M^{15}\,,\,M^{35}]=-M^{13}\\
&[M^{15}\,,\,M^{36}]=0\,,\,[M^{15}\,,\,M^{45}]=-M^{14}\,,\,[M^{15}\,,\,M^{46}]=0\,,\,[M^{15}\,,\,M^{56}]=0\\
&[M^{16}\,,\,M^{23}]=0\,,\,[M^{16}\,,\,M^{24}]=0\,,\,[M^{16}\,,\,M^{25}]=0\,,\,[M^{16}\,,\,M^{26}]=-M^{12}\\
&[M^{16}\,,\,M^{34}]=0\,,\,[M^{16}\,,\,M^{35}]=0\,,\,[M^{16}\,,\,M^{36}]=-M^{13}\,,\,[M^{16}\,,\,M^{45}]=0\\
&[M^{16}\,,\,M^{46}]=-M^{14}\,,\,[M^{16}\,,\,M^{56}]=-M^{15}\,,\,[M^{23}\,,\,M^{24}]=-M^{34}\;,\,[M^{23}\,,\,M^{25}]=-M^{35}\\
&[M^{23}\,,\,M^{26}]=-M^{36}\,,\,[M^{23}\,,\,M^{34}]=M^{24}\,,\,[M^{23}\,,\,M^{35}]=M^{25}\,,\,[M^{23}\,,\,M^{36}]=M^{36}\\
&[M^{23}\,,\,M^{45}]=0\,,\,[M^{23}\,,\,M^{46}]=0\,,\,[M^{23}\,,\,M^{56}]=0\,,\,[M^{24}\,,\,M^{25}]=-M^{45}\\
&[M^{24}\,,\,M^{26}]=-M^{46}\,,\,[M^{24}\,,\,M^{34}]=-M^{23}\,,\,[M^{24}\,,\,M^{35}]=0\,,\,[M^{24}\,,\,M^{36}]=0\\
&[M^{24}\,,\,M^{45}]=M^{25}\,,\,[M^{24}\,,\,M^{46}]=M^{26}\,,\,[M^{24}\,,\,M^{56}]=0\,,\,[M^{25}\,,\,M^{26}]=-M^{56}\\
&[M^{25}\,,\,M^{34}]=0\,,\,[M^{25}\,,\,M^{35}]=-M^{23}\,,\,[M^{25}\,,\,M^{36}]=0\,,\,[M^{25}\,,\,M^{45}]=-M^{24}\\
&[M^{25}\,,\,M^{46}]=0\;,\;[M^{25}\,,\,M^{56}]=0\,,\,[M^{26}\,,\,M^{34}]=0\;,\;[M^{26}\,,\,M^{35}]=0\\
&[M^{26}\,,\,M^{36}]=-M^{23}\,,\,[M^{26}\,,\,M^{45}]=0\,,\,[M^{26}\,,\,M^{46}]=-M^{24}\,,\,[M^{26}\,,\,M^{56}]=-M^{25}\\
&[M^{34}\,,\,M^{35}]=-M^{45}\;,\;[M^{34}\,,\,M^{36}]=-M^{46}\;,\;[M^{34}\;,\;M^{45}]=M^{35}\,,\,[M^{34}\,,\,M^{46}]=M^{36}\\
&[M^{34}\,,\,M^{56}]=0\,.
\end{aligned} \tag{4.84}$$

These fifteen generators M_{ab} $(a<b\,,\,a,b=1,\cdots 6)$ generate the Lie algebra $\mathfrak{so}(6)$ and thus the holonomy of the Hull connection $\tilde{\nabla}^-$ is $SO(6)$ as desired.

4.4 SUSY domain wall metric

The last topic concerns the construction of type II/heterotic supersymmetric solutions on a *compact* six-dimensional space with the Hull connection not being in $SU(3)$. Since the triples obtained in the previous section depend only on x^1 and x^2, we can compactify the x^3, x^4, x^5 and x^6 spaces on T^4 by simply identifying periodically, whereas we consider a periodic array of copies of the solution along the x^1 and x^2 directions.

Let us consider a periodic array of (g, H, φ) [Eqs. (4.47), (4.48), and (4.49)], $(\hat{g}, \hat{H}, \hat{\varphi})$ [eqs. (4.50), (4.51), and (4.52)], or $(\tilde{g}, \tilde{H}, \tilde{\varphi})$ [Eqs. (4.69), (4.70), and (4.71)], which are characterized by a pair of harmonic functions ϕ and ψ. In two dimensions both the real and imaginary parts of any holomorphic function are harmonic. Thus, we can take ϕ to be, say, the real part of any doubly periodic, holomorphic function. In this case, ψ may be taken to be the imaginary part of the same doubly periodic function.

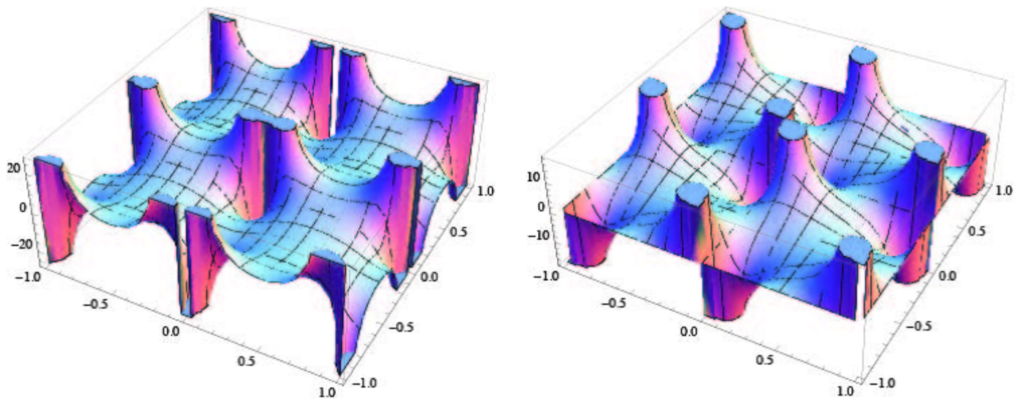

Figure 4.1: The real (left plot) and imaginary (right plot) parts of the $\wp$ function. The fundamental parallelogram can be taken to be $-\frac{1}{2} \leq \frac{x^1}{l} \leq \frac{1}{2}$ and $-\frac{1}{2} \leq \frac{x^2}{l} \leq \frac{1}{2}$.

Since the only nonsingular holomorphic function on T^2 is a constant function, we need to allow some pole singularities in the fundamental parallelogram of the periodic array, which may be seen to be in accordance with the no-go theorems against smooth flux compactifications [50, 51]. The doubly periodic meromorphic functions are known as elliptic functions. It is well known that, for a given periodicity, the field of elliptic functions is generated by Weierstrass' $\wp$ function and its derivative $\wp'$. In the following we consider, as a typical example, the compactification of (g, H, φ), $(\hat{g}, \hat{H}, \hat{\varphi})$ and $(\tilde{g}, \tilde{H}, \tilde{\varphi})$ on a square torus of side l by taking

$$\phi(x^1, x^2) = \mathrm{Re}\ \wp(z), \tag{4.85}$$

$$\psi(x^1, x^2) = \mathrm{Im}\ \wp(z), \tag{4.86}$$

where $\wp(z)$ is of modulus $\tau = i$ or $\tau = e^{\frac{\pi i}{5}}$ and $z = l^{-1}(x^1 + ix^2)$. Our solutions are determined entirely by Weierstrass's $\wp$ function without any reference to α' because of the choice $\mathcal{F} = \mathcal{R}^-$ that causes the rhs of (4.37) to be closed. Note that they solve the heterotic equations of motion up to $\mathcal{O}(\alpha')$.

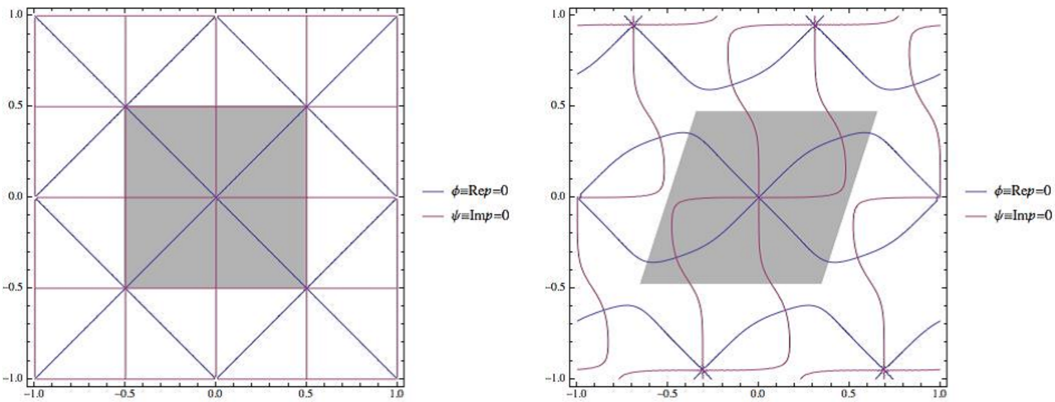

Figure 4.2: The zero loci of the real and imaginary parts of the $\wp$ function for the modulus $\tau = i$ (left plot) and $\tau = e^{\frac{\pi i}{5}}$ (right plot). The shaded region is the fundamental parallelogram.

The real and imaginary parts of $\wp(z)$ are shown in Fig. 4.1. We see that ϕ may take negative as well as positive values, but note that the metric (4.47), (4.50) or (4.50) depends on ϕ through ϕ^2 as we designed, so the solution is only singular where ϕ vanishes (as well as ϕ diverges). Also negative ψ causes no problem as long as ϕ is nonzero.

For any case of (g, H, φ), $(\hat{g}, \hat{H}, \hat{\varphi})$, or $(\tilde{g}, \tilde{H}, \tilde{\varphi})$, some of the components of the metric vanish where $\phi = 0$, and hence the solution is singular. Also, the "string coupling" (= exponential of the dilaton) vanishes there. The $\phi = 0$ curves are shown in Fig. 4.2 for the cases $\tau = i$ and $\tau = e^{\frac{\pi i}{5}}$. For both cases, we see that the fundamental parallelogram (shown by the shaded region) is separated into two distinct smooth regions bordered by the codimension-1 singularity hypersurfaces. The two singularity hypersurfaces intersect at $x^1 = x^2 = 0$, where the $\wp$ function has a unique double pole; its real and imaginary parts rapidly fluctuate at $x^1 = x^2 = 0$. More details about the solution will be reported elsewhere.

4.5 Conclusions

In this chapter, we have shown that two HKT metrics given by $(\Phi, \phi, \boldsymbol{\psi})$ and $(\tilde{\Phi}, \tilde{\phi}, \tilde{\boldsymbol{\psi}})$ can be superposed and lifted to a six-dimensional smeared intersecting solution of type II supergravity if the functions $\Phi, \phi, \tilde{\Phi}$, and $\tilde{\phi}$ are restricted to harmonic functions on the two-dimensional flat space $\mathbf{R^2} = \{(x^1, x^2)\}$, together with $\boldsymbol{\psi} = (0, 0, \psi)$ and $\tilde{\boldsymbol{\psi}} = (0, 0, \tilde{\psi})$ satisfying the Cauchy–Riemann conditions. The simplest geometry that we have considered has an $SO(4)$ ∇^- connection that leads to the $SO(10)$ unbroken gauge symmetry if it is embedded to heterotic string theory as an internal space. By T-duality transformations we have obtained one having an $SO(5)$ or $SO(6)$ ∇^- holonomy. We have also compactified this six-dimensional KT space by taking a periodic array to find a supersymmetric domain wall solution of heterotic supergravity in which the fundamental parallelogram of the two-dimensional periodic array is separated into distinct smooth regions bordered by codimension-1 singularity hypersurfaces.

Chapter 5

Summary

In this thesis, we studied supersymmetric solutions in 6-, 7-, 8-, dimensional supergravity theories. In 7 dimensions, we introduced the class of G_2-structure associated with the Abelian heterotic supergravity theory. The class has a closed 3-form torsion T_7 and an exact Lee form Θ_7. Hence we identified the flux H_7 with T_7 and the dilaton φ_7 with Θ_7, $T_7 = H_7$ and $2d\varphi_7 = \Theta_7$. The class was determined by the choice of (g_7, Ω, F_7), where g_7 was the metric associated with the fundamental 3-form Ω and F_7 was the field strength of $U(1)$ gauge field.In type II supergravity theory, a 3-form flux of the B-field is of closed form; however, the flux H_7 and the field strength F_7 should satisfy the Bianchi identity, $dH_7 = F_7 \wedge F_7$. In addition, F_7 should satisfy the generalized self-dual equation, $*(\Omega \wedge F_7) = F_7$, which arises from G_2 irreducible representation. Thus F_7 was determined algebraically. Then, the defining equations of the class on the cohomogeneity one manifold of the form $\boldsymbol{R}_+ \times S^3 \times S^3$ became first order ordinary differential equations. We obtained the formulae which give the S^3-bolt solutions from regular and Ricci-flat G_2 holonomy metrics, and $T^{1,1}$-bolt solutions were also obtained numerically. In 8 dimensions , similarly to the case of 7 dimensions, we introduced the class of $Spin(7)$-structure associated with the Abelian heterotic supergravity theory. We identified the flux H_8 with the 3-form torsion T_8 and the dilaton φ_8 with the exact Lee form Θ_8. The class was determined by the choice of (g_8, Ψ, F_8), where g_8 is a metric associated with the fundamental 4-form Ψ and F_8 is the field strength of $U(1)$ gauge field. The field strength F_8 was required to satisfy the Bianchi identity and the generalized self-dual equation. We assumed the manifold $\boldsymbol{R}_+ \times M_{3-Sasaki}$, where $M_{3-Sasaki}$ denotes a manifold with a 3-Sasakian structure, and then defining equations reduced to first order ordinary differential equations. Regular solutions with $F_8 \neq 0$ were obtained from regular and Ricci-flat $Spin(7)$ holonomy metrics. It is of great interest to study general solutions, which may shed new light on G_2 or $Spin(7)$ with torsion geometry.

The role of Green–Schwarz mechanism [52] is twofold. First, to cancel the gauge and gravitational anomalies of ten-dimensional type I and heterotic superstring theories, the gauge group must be restricted to $SO(32)$ or $E_8 \times E_8$. Second, the mechanism requires the 2-form B to vary under both the gauge and local Lorentz transformations so that the invariant 3-form field H must be of the form

$$H = dB - \alpha' \left(\omega_{3Y} - \omega_{3L}^{-}\right), \tag{5.1}$$

where ω_{3Y} is the Chern–Simons 3-form associated with the Yang–Mills connection and ω_{3L}^{-} is also a Chern–Simons 3-form. The equation (5.1) leads to the Bianchi identity

$$dH = \alpha' \left(tr\mathcal{F} \wedge \mathcal{F} - tr\mathcal{R}^{-} \wedge \mathcal{R}^{-} \right) . \tag{5.2}$$

The background geometry is restricted in such a way that the curvature 2-form $\mathcal{R}^{-}$ satisfies this identity. The second term in the Bianchi identity (5.2) comes from Chern–Simons ω_{3L}^{-} and $\mathcal{R}^{-}$ arises from the Hull connection ∇^{-} (4.2). Note that the connection (4.2) is different from the one that appears in the supersymmetry variation of the gravitino, $\delta\psi_M \propto \nabla_M^{+}\varepsilon$, where ∇^{+} is called the "Bismut connection" (4.3). The relevance of the difference between the two connections was pointed out by Bergshoeff and de Roo [53], and later emphasized by e.g. [10, 51]. If a field strength is identified with the curvature of a Hull connection, $\mathcal{F} = \mathcal{R}^{-}$, a part of the gauge symmetry $E_8 \times E_8$ is embedded into the background geometry, which is known as standard embedding, especially in the case where $H = 0$ [42]. In the $H = 0$ case, the holonomy of $\nabla^{\pm}$ is in $SU(3)$, and thus, a part of the gauge symmetry is $SU(3)$ by the standard embedding. Therefore, $E_8 \times E_8$ breaks into $E_6 \times E_8$, and this is known as a hallmark of Calabi–Yau compactification. In the $H \neq 0$ case, the holonomy of ∇^{+} belongs to $SU(3)$, whereas the holonomy of ∇^{-} is in $SO(6)$ generally. Hence $E_8 \times E_8$ may break into $SO(10) \times E_8$ by the standard embedding, and this is desirable in a phenomenological view. This is striking contrast to the $H = 0$, Calabi–Yau case. If one doesn't require the identification $\mathcal{F} = \mathcal{R}^{-}$ on the at most-$SO(6)$ Hull connection, then s/he should use the nonstandard embedding that needs complicated mathematical machinery [7, 42].

In dimension 6, we constructed an intersecting metric g_6 by superposing two Gibbons–Hawking metrics with the conformal factors, i.e., HKT metrics. The metric g_6, the fundamental 2-form κ, and the complex $(3,0)$-form Υ satisfy the defining equations of the $SU(3)$-structure associated with the NS–NS sector in type II supergravity, where the Lee form Θ_6 is an exact 1-form and the Bismut torsion T_6 is closed 3-form. Also, the Ricci form vanishes, which gives a necessary condition for the holonomy of the Bismut connection to be contained in $SU(3)$. We identified the flux H_6 with the torsion T_6 and the dilaton φ_6 with Θ_6. Then, we confirmed that two independent Killing spinors existed that satisfy supersymmetric equations in type II supergravity theory. Therefore, the manifold $(M_6, g_6, \kappa, \Upsilon)$ is a Calabi–Yau with torsion manifold, and thus, (g_6, H_6, φ_6) are supersymmetric solutions in the theory [11]. The solution has 2-dimensional harmonic functions ϕ, $\tilde{\phi}$, Φ, and $\tilde{\Phi}$; however, the condition $\phi = \tilde{\phi} = \Phi = \tilde{\Phi}$ is required such that the metric is non-negative and the dilaton takes real value. Thus, the solution is characterized by a pair of harmonic functions (ϕ, ψ), which are related by Cauchy–Riemann equations. If the curvature 2-form of the Hull connection is identified with the gauge field strength in $E_8 \times E_8$ heterotic supergravity theory, then $(g_6, \varphi_6, H_6, \mathcal{F}_6)$ is the solution of the heterotic supergravity theory. Then, the gauge group embedded into E_8 is expected to be at most $SO(6)$, and thus, $SO(10) \subset E_8$ will be obtained. Hence, the supersymmetric solutions should have a holonomy group $SO(6)$ of the Hull connection. However, the holonomy group of the obtained manifold is an $SO(4)$. The metric g_6 has four Killing vectors ∂_3, ∂_4, ∂_5, and ∂_6. To recover the $SO(6)$ holonomy, we T-dualized along ∂_3 and obtained the supersymmetric solution $(\hat{g}_6, \hat{\varphi}, \hat{H})$ having a holonomy group $SO(5)$ of the Hull connection. Since this solution had a Killing vector ∂_5, we further T-dualized along ∂_5. Finally, we obtained the supersymmetric solution $(\tilde{g}_6, \tilde{\varphi}, \tilde{H})$ with the $SO(6)$ holonomy of the Hull connection. To obtain the compact 6-dimensional space, we chose $\phi + i\psi$

as the Weierstrass's $\wp$ function. Then, the solution had codimension-1 singularity hypersurfaces and thus it was interpreted as a "domain wall". It would be interesting to solve the gaugino Dirac equation on this background and compare the spectrum with the corresponding E_8-type supersymmetric nonlinear sigma model [54], similarly to what has been done in the $SU(3)$ ∇^- case [55].

Appendix A

Nijenhuis tensor associated with J

In this section we study the Nijenhuis tensor associated with almost complex structure J. We first introduce an orthonormal basis

$$\begin{aligned}
e^1 &= \sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}dx^1 \ , \ e^2 = \sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}dx^2 \, , \\
e^3 &= \sqrt{\Phi\phi}dx^3 \ , \ e^4 = \sqrt{\frac{\Phi}{\phi}}(dx^4 - \psi dx^3) \, , \\
e^5 &= \sqrt{\tilde{\Phi}\tilde{\phi}}dx^5 \ , \ e^6 = \sqrt{\frac{\tilde{\Phi}}{\tilde{\phi}}}(dx^6 - \tilde{\psi} dx^5)
\end{aligned} \tag{A.1}$$

and their exterior derivatives de^μ $(\mu = 1, \cdots, 6)$ are given by

$$\begin{aligned}
de^1 &= -\frac{1}{\Phi\tilde{\Phi}\phi\tilde{\phi}}\partial_2\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}e^{12} \, , \\
de^2 &= \frac{1}{\Phi\tilde{\Phi}\phi\tilde{\phi}}\partial_1\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}e^{12} \, , \\
de^3 &= \frac{1}{2\Phi\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}[(\phi\partial_1\Phi + \Phi\partial_1\phi)e^{13} + (\phi\partial_2\Phi + \Phi\partial_2\phi)e^{23}] \, , \\
de^4 &= \frac{1}{2\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left\{\left(\partial_1\Phi - \frac{\Phi}{\phi}\partial_1\phi\right)e^{14} + \left(\partial_2\Phi - \frac{\Phi}{\phi}\partial_2\phi\right)e^{24}\right\} - \frac{1}{\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\psi e^{13} + \partial_2\psi e^{23}) \, , \\
de^5 &= \frac{1}{2\tilde{\Phi}\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}[(\tilde{\phi}\partial_1\tilde{\Phi} + \tilde{\Phi}\partial_1\tilde{\phi})e^{15} + (\tilde{\phi}\partial_2\tilde{\Phi} + \tilde{\Phi}\partial_2\tilde{\phi})e^{25}] \, , \\
de^6 &= \frac{1}{2\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left\{\left(\partial_1\tilde{\Phi} - \frac{\tilde{\Phi}}{\tilde{\phi}}\partial_1\tilde{\phi}\right)e^{16} + \left(\partial_2\tilde{\Phi} - \frac{\tilde{\Phi}}{\tilde{\phi}}\partial_2\tilde{\phi}\right)e^{26}\right\} - \frac{1}{\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\tilde{\psi} e^{15} + \partial_2\tilde{\psi} e^{25}) \, .
\end{aligned} \tag{A.2}$$

An almost complex structure $J : T_pM_6 \longrightarrow T_pM_6$ is defined by

$$Je_1 = -\epsilon_1 e_2 \ , \ Je_3 = -\epsilon_2 e_4 \ , \ Je_5 = -\epsilon_3 e_6, \tag{A.3}$$

where e_μ is dual vector for e^μ and $\epsilon_a = \pm 1$ $(a = 1, 2, 3)$. Then the almost complex structure acts on 1-form e^μ as follows:

$$J_* e^1 = \epsilon_1 e^2 \ , \ J_* e^3 = \epsilon_2 e^4 \ , \ J_* e^5 = \epsilon_3 e^6 \, . \tag{A.4}$$

Associated symplectic form κ is defined by

$$\kappa \ := \ \epsilon_1 e^1 \wedge e^2 + \epsilon_2 e^3 \wedge e^4 + \epsilon_3 e^5 \wedge e^6 \, . \tag{A.5}$$

The functions $f^\mu{}_{\nu\rho}$ are defined by

$$de^\mu = \frac{1}{2} f^\mu{}_{\nu\rho} e^{\nu\rho} \tag{A.6}$$

and can be read from (4.22),

$$\begin{aligned}
f^1{}_{12} &= -\frac{1}{\Phi\tilde{\Phi}\phi\tilde{\phi}} \frac{\partial\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}{\partial x^2} = -f^1{}_{21} \ , \ f^2{}_{12} = \frac{1}{\Phi\tilde{\Phi}\phi\tilde{\phi}} \frac{\partial\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}{\partial x^1} = -f^2{}_{21} \, , \\
f^3{}_{13} &= \frac{1}{2\Phi\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} (\phi\partial_1\Phi + \Phi\partial_1\phi) = -f^3{}_{31} \ , \ f^3{}_{23} = \frac{1}{2\Phi\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} (\phi\partial_2\Phi + \Phi\partial_2\phi) = -f^3{}_{32} \, , \\
f^4{}_{13} &= -\frac{1}{\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \partial_1\psi = -f^4{}_{31} \ , \ f^4{}_{14} = \frac{1}{2\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \left(\partial_1\Phi - \frac{\Phi}{\phi}\partial_1\phi\right) = -f^4{}_{41} \, , \\
f^4{}_{23} &= -\frac{1}{\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \partial_2\psi = -f^4{}_{32} \ , \ f^4{}_{24} = \frac{1}{2\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \left(\partial_2\Phi - \frac{\Phi}{\phi}\partial_2\phi\right) = -f^4{}_{42} \, , \\
f^5{}_{15} &= \frac{1}{2\tilde{\Phi}\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} (\tilde{\phi}\partial_1\tilde{\Phi} + \tilde{\Phi}\partial_1\tilde{\phi}) = -f^5{}_{51} \ , \ f^5{}_{25} = \frac{1}{2\tilde{\Phi}\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} (\tilde{\phi}\partial_2\tilde{\Phi} + \tilde{\Phi}\partial_2\tilde{\phi}) = -f^5{}_{52} \, , \\
f^6{}_{15} &= -\frac{1}{\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \partial_1\tilde{\psi} = -f^6{}_{51} \ , \ f^6{}_{16} = \frac{1}{2\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \left(\partial_1\tilde{\Phi} - \frac{\tilde{\Phi}}{\tilde{\phi}}\partial_1\tilde{\phi}\right) = -f^6{}_{61} \, , \\
f^6{}_{25} &= -\frac{1}{\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \partial_2\tilde{\psi} = -f^6{}_{52} \ , \ f^6{}_{26} = \frac{1}{2\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \left(\partial_2\tilde{\Phi} - \frac{\tilde{\Phi}}{\tilde{\phi}}\partial_2\tilde{\phi}\right) = -f^6{}_{62} \, .
\end{aligned} \tag{A.7}$$

Nijenhuis tensor N_J related to an almost complex structure J is defined by

$$N_J(X,Y) := [JX, JY] - [X,Y] - J[X, JY] - J[JX, Y] \, , \tag{A.8}$$

where commutation relation between e_μ is given by

$$[e_\mu, e_\nu] = -f^\rho{}_{\mu\nu} e_\rho \, . \tag{A.9}$$

Each components of the Nijenhuis tensor are the following:

$$\begin{aligned}
N_J(e_1, e_2) &= [Je_1, Je_2] - [e_1, e_2] - J[e_1, Je_2] - J[Je_1, e_2] = [-\epsilon_1 e_2, \epsilon_1 e_1] - [e_1, e_2] \\
&= 0 \, ,
\end{aligned}$$

$$
\begin{aligned}
N_J(e_1,e_3) &= [Je_1,Je_3]-[e_1,e_3]-J[e_1,Je_3]-J[Je_1,e_3]\\
&= [-\epsilon_1 e_2,-\epsilon_2 e_4]-[e_1,e_3]-J[e_1,-\epsilon_2 e_4]-J[-\epsilon_1 e_2,e_3]\\
&= -\epsilon_1\epsilon_2 f^{\mu}{}_{24}e_\mu+f^{\mu}{}_{13}e_\mu-\epsilon_2 f^{\mu}{}_{14}Je_\mu-\epsilon_1 f^{\mu}{}_{23}Je_\mu\\
&= -\epsilon_1\epsilon_2 f^{4}{}_{24}e_4+f^{3}{}_{13}e_3+f^{4}{}_{13}e_4-\epsilon_2 f^{4}{}_{14}Je_4-\epsilon_1 f^{3}{}_{23}Je_3-\epsilon_1 f^{4}{}_{23}Je_4\\
&= (f^{3}{}_{13}-f^{4}{}_{14}-\epsilon_1\epsilon_2 f^{4}{}_{23})e_3+(\epsilon_1\epsilon_2 f^{3}{}_{23}-\epsilon_1\epsilon_2 f^{4}{}_{24}+f^{4}{}_{13})e_4\\
&= \frac{1}{\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\partial_1\phi+\epsilon_1\epsilon_2\partial_2\psi\right)e_3+\frac{1}{\phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left(\epsilon_1\epsilon_2\partial_2\phi-\partial_1\psi\right)e_4\,.
\end{aligned}
$$

If $\epsilon_1\epsilon_2=-1$, $N_J(e_1,e_3)=0$ by Cauchy-Riemann equation (4.20). Therefore if we require $N_J=0$, relative sign of ϵ_1 and ϵ_2 is determined by the following relation,

$$
\epsilon_1\epsilon_2=-1\,. \tag{A.10}
$$

The reminder of this sections, we impose ϵ_1 and ϵ_2 on $\epsilon_1\epsilon_2=-1$.

$$
\begin{aligned}
N_J(e_1,e_4) &= [Je_1,Je_4]-[e_1,e_4]-J[e_1,Je_4]-J[Je_1,e_4]\\
&= [-\epsilon_1 e_2,\epsilon_2 e_3]-[e_1,e_4]-J[e_1,\epsilon_2 e_3]-J[-\epsilon_1 e_2,e_4]\\
&= \epsilon_1\epsilon_2 f^{\mu}{}_{23}e_\mu+f^{\mu}{}_{14}e_\mu+\epsilon_2 f^{\mu}{}_{13}Je_\mu-\epsilon_1 f^{\mu}{}_{24}Je_\mu\\
&= \epsilon_1\epsilon_2 f^{3}{}_{23}e_3+\epsilon_1\epsilon_2 f^{4}{}_{23}e_4+f^{4}{}_{14}e_4+\epsilon_2 f^{3}{}_{13}Je_3+\epsilon_2 f^{4}{}_{13}Je_4-\epsilon_1 f^{4}{}_{24}Je_4\\
&= (\epsilon_1\epsilon_2 f^{3}{}_{23}+f^{4}{}_{13}-\epsilon_1\epsilon_2 f^{4}{}_{24})e_3+(\epsilon_1\epsilon_2 f^{4}{}_{23}+f^{4}{}_{14}-f^{3}{}_{13})e_4\\
&= (-f^{3}{}_{23}+f^{4}{}_{13}+f^{4}{}_{24})e_3+(-f^{4}{}_{23}+f^{4}{}_{14}+f^{3}{}_{13})e_4\,.\\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
N_J(e_1,e_5) &= [Je_1,Je_5]-[e_1,e_5]-J[e_1,Je_5]-J[Je_1,e_5]\\
&= [-\epsilon_1 e_2,-\epsilon_3 e_6]-[e_1,e_5]-J[e_1,-\epsilon_3 e_6]-J[-\epsilon_1 e_2,e_5]\\
&= -\epsilon_1\epsilon_3 f^{\mu}{}_{26}e_\mu+f^{\mu}{}_{15}e_\mu-\epsilon_3 f^{\mu}{}_{16}Je_\mu-\epsilon_1 f^{\mu}{}_{25}Je_\mu\\
&= -\epsilon_1\epsilon_3 f^{6}{}_{26}e_6+f^{5}{}_{15}e_5+f^{6}{}_{15}e_6-\epsilon_3 f^{6}{}_{16}Je_6-\epsilon_1 f^{5}{}_{25}Je_5-\epsilon_1 f^{6}{}_{25}Je_6\\
&= (f^{5}{}_{15}-f^{6}{}_{16}-\epsilon_1\epsilon_3 f^{6}{}_{25})e_5+(f^{6}{}_{15}+\epsilon_1\epsilon_3 f^{5}{}_{25}-\epsilon_1\epsilon_3 f^{6}{}_{26})e_6\\
&= \frac{1}{\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\tilde{\phi}+\epsilon_1\epsilon_3\partial_2\tilde{\psi})e_5+\frac{1}{\tilde{\phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(-\partial_1\tilde{\psi}+\epsilon_1\epsilon_3\partial_2\tilde{\phi})e_6\,.
\end{aligned}
$$

If $\epsilon_1\epsilon_3=-1$, $N_J(e_1,e_5)=0$ by Cauchy-Riemann equation (4.20). Therefore if we require $N_J=0$, relative sign of ϵ_1 and ϵ_3 is determined by the following relation,

$$
\epsilon_1\epsilon_3=-1\,. \tag{A.11}
$$

The rest of this sections, we impose ϵ_1 and ϵ_3 on $\epsilon_1\epsilon_3=-1$.

$$
\begin{aligned}
N_J(e_1,e_6) &= [Je_1,Je_6]-[e_1,e_6]-J[e_1,Je_6]-J[Je_1,e_6]\\
&= (\epsilon_1\epsilon_3 f^{5}{}_{25}+f^{6}{}_{15}-\epsilon_1\epsilon_3 f^{6}{}_{26})e_5+(\epsilon_1\epsilon_3 f^{6}{}_{25}+f^{6}{}_{16}-f^{5}{}_{15})e_6\,.\\
&= 0\,.
\end{aligned}
$$

By similar calculation or $Je_1 = -\epsilon_1 e_2$, $Je_2 = \epsilon_1 e_1$, we can see immediately

$$N_J(e_2, e_\mu) = 0\,,$$

$$\begin{aligned} N_J(e_3, e_4) &= [Je_3, Je_4] - [e_3, e_4] - J[e_3, Je_4] - J[Je_3, e_4] \\ &= [-\epsilon_2 e_4, \epsilon_2 e_3] - [e_3, e_4] \\ &= 0\,, \end{aligned}$$

$$\begin{aligned} N_J(e_3, e_5) &= [Je_3, Je_5] - [e_3, e_5] - J[e_3, Je_5] - J[Je_3, e_5] \\ &= [-\epsilon_2 e_4, -\epsilon_3 e_6] - [e_3, e_5] - J[e_3, -\epsilon_3 e_6] - J[-\epsilon_2 e_4, e_5] \\ &= -\epsilon_2 \epsilon_3 f^\mu{}_{46} e_\mu + f^\mu{}_{35} e_\mu - \epsilon_3 f^\mu{}_{36} J e_\mu - \epsilon_2 f^\mu{}_{45} J e_\mu \\ &= 0\,, \end{aligned}$$

$$\begin{aligned} N_J(e_3, e_6) &= [Je_3, Je_6] - [e_3, e_6] - J[e_3, Je_6] - J[Je_3, e_6] \\ &= [-\epsilon_2 e_4, \epsilon_3 e_5] - [e_3, e_6] - J[e_3, \epsilon_3 e_5] - J[-\epsilon_2 e_4, e_6] \\ &= 0\,, \end{aligned}$$

By similar calculation or $Je_3 = -\epsilon_2 e_4$, $Je_4 = \epsilon_2 e_3$, we can see immediately

$$N_J(e_4, e_\mu) = 0.$$

$$\begin{aligned} N_J(e_5, e_6) &= [Je_5, Je_6] - [e_5, e_6] - J[e_5, Je_6] - J[Je_5, e_6] \\ &= [-\epsilon_3 e_6, \epsilon_3 e_5] - [e_5, e_6] \\ &= 0 \end{aligned}$$

Consequently under conditions $\epsilon_1 \epsilon_2 = -1$ and $\epsilon_1 \epsilon_3 = -1$ [1] Nijenhuis tensor is vanish,

$$N_J = 0\,. \tag{A.12}$$

Then almost complex structure J is also complex structure.

[1] A condition for $\epsilon_2 \epsilon_3$ can be obtained from $\epsilon_1 \epsilon_2 = -1$ and $\epsilon_1 \epsilon_3 = -1$, $\epsilon_2 \epsilon_3 = 1$.

Appendix B

$(1,0)$**-form**

We introduce complex 1-form $\zeta^a, \bar{\zeta}^{\bar{a}}$ by the following,

$$\zeta^1 = \sqrt{\phi\tilde{\phi}}(dx^1 + idx^2)\ ,\ \zeta^2 = \sqrt{\phi}dx^3 - i\frac{1}{\sqrt{\phi}}(dx^4 - \psi dx^3)\ ,\ \zeta^3 = \sqrt{\tilde{\phi}}dx^5 - i\frac{1}{\sqrt{\tilde{\phi}}}(dx^6 - \tilde{\psi}dx^5)\ ,$$
$$\bar{\zeta}^{\bar{1}} = \sqrt{\phi\tilde{\phi}}(dx^1 - idx^2)\ ,\ \bar{\zeta}^{\bar{2}} = \sqrt{\phi}dx^3 + i\frac{1}{\sqrt{\phi}}(dx^4 - \psi dx^3)\ ,\ \bar{\zeta}^{\bar{3}} = \sqrt{\tilde{\phi}}dx^5 + i\frac{1}{\sqrt{\tilde{\phi}}}(dx^6 - \tilde{\psi}dx^5)\ . \tag{B.1}$$

Then the dual vector fields are

$$\zeta_1 = \frac{1}{2\sqrt{\phi\tilde{\phi}}}\left(\frac{\partial}{\partial x^1} - i\frac{\partial}{\partial x^2}\right)\ ,\ \bar{\zeta}_{\bar{1}} = \frac{1}{2\sqrt{\phi\tilde{\phi}}}\left(\frac{\partial}{\partial x^1} + i\frac{\partial}{\partial x^2}\right)\ ,$$
$$\zeta_2 = \frac{1}{2}\left\{\frac{1}{\sqrt{\phi}}\frac{\partial}{\partial x^3} + i\left(\sqrt{\phi} - i\frac{\psi}{\sqrt{\phi}}\right)\frac{\partial}{\partial x^4}\right\}\ ,\ \bar{\zeta}_{\bar{2}} = \frac{1}{2}\left\{\frac{1}{\sqrt{\phi}}\frac{\partial}{\partial x^3} - i\left(\sqrt{\phi} + i\frac{\psi}{\sqrt{\phi}}\right)\frac{\partial}{\partial x^4}\right\}\ ,$$
$$\zeta_3 = \frac{1}{2}\left\{\frac{1}{\sqrt{\tilde{\phi}}}\frac{\partial}{\partial x^3} + i\left(\sqrt{\tilde{\phi}} - i\frac{\tilde{\psi}}{\sqrt{\tilde{\phi}}}\right)\frac{\partial}{\partial x^4}\right\}\ ,\ \bar{\zeta}_{\bar{3}} = \frac{1}{2}\left\{\frac{1}{\sqrt{\tilde{\phi}}}\frac{\partial}{\partial x^3} - i\left(\sqrt{\tilde{\phi}} + i\frac{\tilde{\psi}}{\sqrt{\tilde{\phi}}}\right)\frac{\partial}{\partial x^4}\right\}\ . \tag{B.2}$$

Using $J_*e^1 = \epsilon_1 e^2$, $J_*e^3 = \epsilon_2 e^4$, $J_*e^5 = \epsilon_3 e^6$ and the definition of e^a, we can obtain J_*dx^μ and $J_*\zeta^a$ as follows: $J_*dx^\mu\ (\mu = 1, \cdots, 6)$ are

$$J_*dx^1 = \epsilon_2 dx^2\ ,\ J_*dx^2 = -\epsilon_2 dx^1\ ,$$
$$J_*dx^3 = \frac{\epsilon_2}{\phi}(dx^4 - \psi dx^3)\ ,\ J_*dx^4 = \epsilon_2\left\{\frac{\psi}{\phi}dx^4 - \phi\left(1 + \frac{\psi^2}{\phi^2}\right)dx^3\right\}\ ,$$
$$J_*dx^5 = \frac{\epsilon_3}{\tilde{\phi}}(dx^6 - \tilde{\psi}dx^5)\ ,\ J_*dx^6 = \epsilon_3\left\{\frac{\tilde{\psi}}{\tilde{\phi}}dx^6 - \tilde{\phi}\left(1 + \frac{\tilde{\psi}^2}{\tilde{\phi}^2}\right)dx^5\right\} \tag{B.3}$$

and $J_*\zeta^a\ (a = 1, \cdots, 6)$ are

$$J_*\zeta^1 = -i\epsilon_1\zeta^1\ ,\ J_*\zeta^2 = i\epsilon_2\zeta^2\ ,\ J_*\zeta^3 = i\epsilon_3\zeta^3\ . \tag{B.4}$$

When $\epsilon_1 = -1$ together with $\epsilon_2 = 1$ and $\epsilon_3 = 1$, ζ^1, ζ^2 and ζ^3 are (1,0)-form [56].

Let us rewrite fundamental 2-form κ,

$$\kappa = \epsilon_1 \Phi\tilde{\Phi}\phi\tilde{\phi} dx^{12} + \epsilon_2 \Phi dx^{34} + \epsilon_3 \tilde{\Phi} dx^{56} \,. \tag{B.5}$$

By using ζ^a and $\bar{\zeta}^{\bar{a}}$, $dx^\mu \, (\mu = 1, \cdots, 6)$ are given by

$$\begin{aligned}
dx^1 &= \frac{1}{2\sqrt{\phi\tilde{\phi}}}(\zeta^1 + \bar{\zeta}^{\bar{1}}) \,,\; dx^2 = \frac{1}{2i\sqrt{\phi\tilde{\phi}}}(\zeta^1 - \bar{\zeta}^{\bar{1}}) \,, \\
dx^3 &= \frac{1}{2\sqrt{\phi}}(\zeta^2 + \bar{\zeta}^{\bar{2}}) \,,\; dx^4 = -\frac{\sqrt{\phi}}{2i}(\zeta^2 - \bar{\zeta}^{\bar{2}}) + \frac{\psi}{2\sqrt{\phi}}(\zeta^2 + \bar{\zeta}^{\bar{2}}) \,, \\
dx^5 &= \frac{1}{2\sqrt{\tilde{\phi}}}(\zeta^3 + \bar{\zeta}^{\bar{3}}) \,,\; dx^6 = -\frac{\sqrt{\tilde{\phi}}}{2i}(\zeta^3 - \bar{\zeta}^{\bar{3}}) + \frac{\tilde{\psi}}{2\sqrt{\tilde{\phi}}}(\zeta^3 + \bar{\zeta}^{\bar{3}}) \,.
\end{aligned} \tag{B.6}$$

Also $dx^{\mu\nu}$ are calculated as follows:

$$\begin{aligned}
dx^{12} &= \frac{i}{2\phi\tilde{\phi}}\zeta^1\bar{\zeta}^{\bar{1}} \,,\; dx^{34} = -\frac{i}{2}\zeta^2\bar{\zeta}^{\bar{2}} \,,\; dx^{56} = -\frac{i}{2}\zeta^3\bar{\zeta}^{\bar{3}} \,, \\
dx^{13} &= \frac{1}{4\phi\sqrt{\tilde{\phi}}}(\zeta^1\zeta^2 + \zeta^1\bar{\zeta}^{\bar{2}} + \bar{\zeta}^{\bar{1}}\zeta^2 + \bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{2}}) \,, \\
dx^{14} &= \frac{1}{4\sqrt{\tilde{\phi}}}\left[(i + \frac{\psi}{\phi})\zeta^1\zeta^2 + (-i + \frac{\psi}{\phi})\zeta^1\bar{\zeta}^{\bar{2}} + (i + \frac{\psi}{\phi})\bar{\zeta}^{\bar{1}}\zeta^2 + (-i + \frac{\psi}{\phi})\bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{2}}\right] \,, \\
dx^{15} &= \frac{1}{4\tilde{\phi}\sqrt{\phi}}(\zeta^1\zeta^3 + \zeta^1\bar{\zeta}^{\bar{3}} + \bar{\zeta}^{\bar{1}}\zeta^3 + \bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{3}}) \,, \\
dx^{16} &= \frac{1}{4\sqrt{\phi}}\left[\left(i + \frac{\tilde{\psi}}{\tilde{\phi}}\right)\zeta^1\zeta^3 + \left(-i + \frac{\tilde{\psi}}{\tilde{\phi}}\right)\zeta^1\bar{\zeta}^{\bar{3}} + \left(i + \frac{\tilde{\psi}}{\tilde{\phi}}\right)\bar{\zeta}^{\bar{1}}\zeta^3 + \left(-i + \frac{\tilde{\psi}}{\tilde{\phi}}\right)\bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{3}}\right] \,, \\
dx^{23} &= -\frac{i}{4\phi\sqrt{\tilde{\phi}}}(\zeta^1\zeta^2 + \zeta^1\bar{\zeta}^{\bar{2}} - \bar{\zeta}^{\bar{1}}\zeta^2 - \bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{2}}) \,, \\
dx^{24} &= \frac{1}{4\sqrt{\tilde{\phi}}}\left[\left(1 - i\frac{\psi}{\phi}\right)\zeta^1\zeta^2 - \left(1 + i\frac{\psi}{\phi}\right)\zeta^1\bar{\zeta}^{\bar{2}} + \left(-1 + i\frac{\psi}{\phi}\right)\bar{\zeta}^{\bar{1}}\zeta^2 + \left(1 + i\frac{\psi}{\phi}\right)\bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{2}}\right] \,, \\
dx^{25} &= -\frac{i}{4\tilde{\phi}\sqrt{\phi}}(\zeta^1\zeta^3 + \zeta^1\bar{\zeta}^{\bar{3}} - \bar{\zeta}^{\bar{1}}\zeta^3 - \bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{3}}) \,, \\
dx^{26} &= \frac{1}{4\sqrt{\phi}}\left[\left(1 - i\frac{\tilde{\psi}}{\tilde{\phi}}\right)\zeta^1\zeta^3 - \left(1 + i\frac{\tilde{\psi}}{\tilde{\phi}}\right)\zeta^1\bar{\zeta}^{\bar{3}} + \left(-1 + i\frac{\tilde{\psi}}{\tilde{\phi}}\right)\bar{\zeta}^{\bar{1}}\zeta^3 + \left(1 + i\frac{\tilde{\psi}}{\tilde{\phi}}\right)\bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{3}}\right]
\end{aligned}$$

and then $d\zeta^a$ and $d\bar{\zeta}^{\bar{a}}$ are given by

$$\begin{aligned}
d\zeta^1 &= -\frac{1}{4\sqrt{\phi\tilde{\phi}}}\{\partial_1 \log \phi\tilde{\phi} + i\partial_2 \log \phi\tilde{\phi}\}\zeta^1\bar{\zeta}^{\bar{1}}\,, \\
d\bar{\zeta}^{\bar{1}} &= \frac{1}{4\sqrt{\phi\tilde{\phi}}}\{\partial_1 \log \phi\tilde{\phi} - i\partial_2 \log \phi\tilde{\phi}\}\zeta^1\bar{\zeta}^{\bar{1}}\,, \\
d\zeta^2 &= \frac{1}{4\sqrt{\phi\tilde{\phi}}}\Big[(\partial_1 \log \phi - i\partial_2 \log \phi)\zeta^1\zeta^2 + 2(\partial_1 \log \phi - i\partial_2 \log \phi)\zeta^1\bar{\zeta}^{\bar{2}} - (\partial_1 \log \phi + i\partial_2 \log \phi)\bar{\zeta}^{\bar{1}}\zeta^2\Big]\,, \\
d\bar{\zeta}^{\bar{2}} &= \frac{1}{4\sqrt{\phi\tilde{\phi}}}\Big[(\partial_1 \log \phi + i\partial_2 \log \phi)\bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{2}} - 2(\partial_1 \log \phi + i\partial_2 \log \phi)\zeta^2\bar{\zeta}^{\bar{1}} - (\partial_1 \log \phi - i\partial_2 \log \phi)\zeta^1\bar{\zeta}^{\bar{2}}\Big]\,, \\
d\zeta^3 &= \frac{1}{4\sqrt{\phi\tilde{\phi}}}\Big[(\partial_1 \log \tilde{\phi} - i\partial_2 \log \tilde{\phi})\zeta^1\zeta^3 + 2(\partial_1 \log \tilde{\phi} - i\partial_2 \log \tilde{\phi})\zeta^1\bar{\zeta}^{\bar{3}} - (\partial_1 \log \tilde{\phi} + i\partial_2 \log \tilde{\phi})\bar{\zeta}^{\bar{1}}\zeta^3\Big]\,, \\
d\bar{\zeta}^{\bar{3}} &= \frac{1}{4\sqrt{\phi\tilde{\phi}}}\Big[(\partial_1 \log \tilde{\phi} + i\partial_2 \log \tilde{\phi})\bar{\zeta}^{\bar{1}}\bar{\zeta}^{\bar{3}} - 2(\partial_1 \log \tilde{\phi} + i\partial_2 \log \tilde{\phi})\zeta^3\bar{\zeta}^{\bar{1}} - (\partial_1 \log \tilde{\phi} - i\partial_2 \log \tilde{\phi})\zeta^1\bar{\zeta}^{\bar{3}}\Big]\,.
\end{aligned} \tag{B.7}$$

Therefore we have κ expressed as (1,1)-form,

$$\kappa = \frac{i}{2}\{\epsilon_1\Phi\tilde{\Phi}\zeta^1\bar{\zeta}^{\bar{1}} - \epsilon_2\Phi\zeta^2\bar{\zeta}^{\bar{2}} - \epsilon_3\tilde{\Phi}\zeta^3\bar{\zeta}^{\bar{3}}\}\,. \tag{B.8}$$

We consider the Bismut torsion $T = \epsilon_4 J_* d\kappa$. The exterior derivative of κ is given by

$$\begin{aligned}
d\kappa &= -\frac{i}{4\sqrt{\phi\tilde{\phi}}}\Big[\epsilon_2\Big\{(\partial_1\Phi - i\partial_2\Phi)\zeta^{12}\bar{\zeta}^{\bar{2}} - (\partial_1\Phi + i\partial_2\Phi)\zeta^2\bar{\zeta}^{\bar{1}\bar{2}}\Big\} \\
&\qquad +\epsilon_3\Big\{(\partial_1\tilde{\Phi} - i\partial_2\tilde{\Phi})\zeta^{13}\bar{\zeta}^{\bar{3}} - (\partial_1\tilde{\Phi} + i\partial_2\tilde{\Phi})\zeta^3\bar{\zeta}^{\bar{1}\bar{3}}\Big\}\Big]
\end{aligned}$$

and thus we have

$$\begin{aligned}
J_* d\kappa &= -\frac{i}{4\sqrt{\phi\tilde{\phi}}}\Big[\epsilon_2\Big\{-i\epsilon_1(\partial_1\Phi - i\partial_2\Phi)\zeta^{12}\bar{\zeta}^{\bar{2}} - i\epsilon_1(\partial_1\Phi + i\partial_2\Phi)\zeta^2\bar{\zeta}^{\bar{1}\bar{2}}\Big\} \\
&\qquad +\epsilon_3\Big\{-i\epsilon_1(\partial_1\tilde{\Phi} - i\partial_2\tilde{\Phi})\zeta^{13}\bar{\zeta}^{\bar{3}} - i\epsilon_1(\partial_1\tilde{\Phi} + i\partial_2\tilde{\Phi})\zeta^3\bar{\zeta}^{\bar{1}\bar{3}}\Big\}\Big] \\
&= \frac{1}{4\sqrt{\phi\tilde{\phi}}}\Big[\Big\{(\partial_1\Phi - i\partial_2\Phi)\zeta^{12}\bar{\zeta}^{\bar{2}} + (\partial_1\Phi + i\partial_2\Phi)\zeta^2\bar{\zeta}^{\bar{1}\bar{2}}\Big\} + \Big\{(\partial_1\tilde{\Phi} - i\partial_2\tilde{\Phi})\zeta^{13}\bar{\zeta}^{\bar{3}} + (\partial_1\tilde{\Phi} + i\partial_2\tilde{\Phi})\zeta^3\bar{\zeta}^{\bar{1}\bar{3}}\Big\}\Big] \\
&= \frac{1}{4\sqrt{\phi\tilde{\phi}}}\Big[(\partial_1\Phi)(\zeta^1 - \bar{\zeta}^{\bar{1}})\zeta^2\bar{\zeta}^{\bar{2}} - i(\partial_2\Phi)(\zeta^1 + \bar{\zeta}^{\bar{1}})\zeta^2\bar{\zeta}^{\bar{2}} + (\partial_1\tilde{\Phi})(\zeta^1 - \bar{\zeta}^{\bar{1}})\zeta^3\bar{\zeta}^{\bar{3}} - i(\partial_2\tilde{\Phi})(\zeta^1 + \bar{\zeta}^{\bar{1}})\zeta^3\bar{\zeta}^{\bar{3}}\Big] \\
&= (\partial_2\Phi)dx^{134} - (\partial_1\Phi)dx^{234} + (\partial_2\tilde{\Phi})dx^{156} - (\partial_1\tilde{\Phi})dx^{256}\,.
\end{aligned}$$

Indeed, this is coincident with the Bismut torsion (4.26) under $\epsilon_4 = -1$.

Appendix C

The $SU(3)$-structure on the superposed HKT metrics

The necessary and sufficient conditions for preservation of supersymmetry in type II supergravity are that the manifold M_6 has an SU(3) structure satisfying the differential conditions [9],

$$d(e^{-2\varphi}\Upsilon) = 0 \tag{C.1}$$

$$d(e^{-2\varphi} * \kappa) = 0 \tag{C.2}$$

$$T = -e^{2\varphi} * d(e^{-2\varphi}\kappa)\,, \tag{C.3}$$

where φ is dilaton and Υ is complex (3,0)-form. In this appendix, we check the manifold (M_6, g, J) admits the $SU(3)$-structure, where the metric g is given by (4.19) and the complex structure J is defined by (4.23).

Lee form Θ is defined by

$$\Theta(X) = -\frac{1}{2}\sum_{\mu=1}^{6} T(JX, e_\mu, Je_\mu)\,, \tag{C.4}$$

where X is an vector field on TM_6. From (4.23), an complex structure $J : T_pM_6 \longrightarrow T_pM_6$ is given by

$$Je_1 = -\epsilon_1 e_2\ ,\ Je_3 = -\epsilon_2 e_4\ ,\ Je_5 = -\epsilon_3 e_6, \tag{C.5}$$

where e_μ is dual vector for e^μ and $\epsilon_a = \pm 1$ $(a = 1, 2, 3)$. Then, $\Theta(X)$ is

$$\Theta(X) \ = \ \epsilon_1 T(JX, e_1, e_2) + \epsilon_2 T(JX, e_3, e_4) + \epsilon_3 T(JX, e_5, e_6)\,. \tag{C.6}$$

Hence components of the Lee form θ in the orthonormal base $\{e_\mu\}$ are the following,

$$\begin{aligned}\Theta(e_1) \ &= \ -\epsilon_1\epsilon_2 T(e_2, e_3, e_4) - \epsilon_1\epsilon_3 T(e_2, e_5, e_6)\,,\\ &= \ -\frac{\epsilon_4}{\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\partial_1\Phi - \frac{\epsilon_4}{\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\partial_1\tilde{\Phi}\,,\end{aligned}$$

$$\begin{aligned}\Theta(e_2) \ &= \ \epsilon_1\epsilon_2 T(e_1, e_3, e_4) + \epsilon_1\epsilon_3 T(e_1, e_5, e_6)\,,\\ &= \ -\frac{\epsilon_4}{\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\partial_2\Phi - \frac{\epsilon_4}{\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\partial_2\tilde{\Phi}\,,\end{aligned}$$

$$
\begin{aligned}
\Theta(e_3) &= -\epsilon_1\epsilon_2 T(e_4, e_1, e_2) - \epsilon_2\epsilon_3 T(e_4, e_5, e_6)\,,\\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\Theta(e_4) &= \epsilon_1\epsilon_2 T(e_3, e_1, e_2) + \epsilon_2\epsilon_3 T(e_3, e_5, e_6)\,,\\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\Theta(e_5) &= -\epsilon_1\epsilon_3 T(e_6, e_1, e_2) - \epsilon_2\epsilon_3 T(e_6, e_3, e_4)\,,\\
&= 0\,,
\end{aligned}
$$

$$
\begin{aligned}
\Theta(e_6) &= \epsilon_1\epsilon_3 T(e_5, e_1, e_2) + \epsilon_2\epsilon_3 T(e_5, e_3, e_4)\,,\\
&= 0\,.
\end{aligned}
$$

Therefore the Lee form Θ is

$$
\Theta = -\frac{\epsilon_4}{\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left[\left(\frac{1}{\Phi}\partial_1\Phi + \frac{1}{\tilde{\Phi}}\partial_1\tilde{\Phi}\right)e^1 + \left(\frac{1}{\Phi}\partial_2\Phi + \frac{1}{\tilde{\Phi}}\partial_2\tilde{\Phi}\right)e^2\right]\,. \tag{C.7}
$$

Then it should be noticed that Θ can be rewritten as an exact form ($d\Theta = 0$);

$$
\Theta = -\epsilon_4\left(d\log\Phi + d\log\tilde{\Phi}\right) = -\epsilon_4 d\log\Phi\tilde{\Phi}\,. \tag{C.8}
$$

When we identify dilaton with Lee form,

$$
d\varphi = -\frac{1}{2}\epsilon_4\Theta\,, \tag{C.9}
$$

the dilaton φ is related to Θ under $\epsilon_4 = -1$,

$$
\Theta = 2d\varphi\,,\ \varphi = \log\sqrt{\Phi\tilde{\Phi}}\,. \tag{C.10}
$$

The Bismut torsion T is given by

$$
T = \epsilon_4 J_* d\kappa \tag{C.11}
$$

and the torsion T satisfies (C.3) under the condition $\epsilon_4 = -1$ as follows: Firstly we consider (C.3). The exterior derivative of κ is

$$
d\kappa = \frac{\epsilon_2}{\Phi\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\Phi e^{134} + \partial_2\Phi e^{234}) + \frac{\epsilon_3}{\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}(\partial_1\tilde{\Phi}e^{156} + \partial_2\tilde{\Phi}e^{256})
$$

and thus

$$
\begin{aligned}
d(e^{-2\varphi}\kappa) &= d(\frac{1}{\Phi\tilde{\Phi}}\kappa)\,,\\
&= \frac{1}{\Phi\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left[-\epsilon_2(\partial_1\log\tilde{\Phi})e^{134} - \epsilon_2(\partial_2\log\tilde{\Phi})e^{234} - \epsilon_3(\partial_2\log\Phi)e^{156} - \epsilon_3(\partial_2\log\Phi)e^{256}\right]\,.
\end{aligned}
$$

The Hodge dual associated with (4.19) of e^{134}, e^{234}, e^{156} and e^{256} are given by

$$
\begin{aligned}
&*e^{134} = \iota_{e_4}\iota_{e_3}\iota_{e_1}e^{123456} = e^{256}\,, \; *e^{234} = \iota_{e_4}\iota_{e_3}\iota_{e_2}e^{123456} = -e^{156}\,,\\
&*e^{156} = \iota_{e_6}\iota_{e_5}\iota_{e_1}e^{123456} = e^{234}\,, \; *e^{256} = \iota_{e_6}\iota_{e_5}\iota_{e_2}e^{123456} = -e^{134}\,,
\end{aligned}
\tag{C.12}
$$

and thus the torsion T in (C.3) is the following,

$$
T = -\frac{\epsilon_2}{\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}[\{(\partial_2 \log \Phi)e^{134} - (\partial_2 \log \Phi)e^{234}\} + \{(\partial_2 \log \tilde{\Phi})e^{156} - (\partial_1 \log \tilde{\Phi})e^{256}\}] \quad (\because \epsilon_2 = \epsilon_3) \tag{C.13}
$$

Secondly, we consider (C.11). $J_* e^{abc}$ are given by

$$
\begin{aligned}
J_*e^{134} &= J_*e^1 \wedge J_*e^3 \wedge J_*e^4 = -\epsilon_1 e^{243} = \epsilon_1 e^{234}\,,\\
J_*e^{234} &= J_*e^2 \wedge J_*e^3 \wedge J_*e^4 = \epsilon_1 e^{143} = -\epsilon_1 e^{134}\,,\\
J_*e^{156} &= J_*e^1 \wedge J_*e^5 \wedge J_*e^6 = -\epsilon_1 e^{265} = \epsilon_1 e^{256}\,,\\
J_*e^{256} &= J_*e^2 \wedge J_*e^5 \wedge J_*e^6 = \epsilon_1 e^{165} = -\epsilon_1 e^{156}\,,
\end{aligned}
$$

and then the torsion T in (C.11) takes the form

$$
\begin{aligned}
T &= -\frac{\epsilon_4}{\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left\{-\frac{1}{\Phi}(\partial_2\Phi e^{134} - \partial_1\Phi e^{234}) - \frac{1}{\tilde{\Phi}}(\partial_2\tilde{\Phi}e^{156} - \partial_1\tilde{\Phi}e^{256})\right\}\,,\\
&= -\epsilon_4\{-(\partial_2\Phi dx^{134} - \partial_1\Phi dx^{234}) - (\partial_2\tilde{\Phi}dx^{156} - \partial_1\tilde{\Phi}dx^{256})\}\,.
\end{aligned}
\tag{C.14}
$$

Thus, the Bismut torsion (C.11) coincides with (C.14) when $\epsilon_4 = -1$ and $\epsilon_2 = 1$. In addition, we calculate dT,

$$
dT = \epsilon_1\epsilon_4\{\epsilon_2(\partial_2^2\Phi + \partial_1^2\Phi)dx^{1234} + \epsilon_3(\partial_2^2\tilde{\Phi} + \partial_1^2\tilde{\Phi})dx^{1256}\}\,. \tag{C.15}
$$

Namely, the 3-form torsion T is closed form, $dT = 0$, if Φ and $\tilde{\Phi}$ are harmonic function on $\boldsymbol{R}^2$.

We consider (C.2). The Hodge dual associated with (4.19) of the fundamental 2-form κ is given by

$$
\begin{aligned}
*\kappa &= \epsilon_1 e^{3456} + \epsilon_2 e^{1256} + \epsilon_3 e^{1234}\,,\\
&= \epsilon_1\Phi\tilde{\Phi}dx^{3456} + \epsilon_2\Phi\tilde{\Phi}^2\phi\tilde{\phi}dx^{1256} + \epsilon_3\Phi^2\tilde{\Phi}\phi\tilde{\phi}dx^{1234}\,.
\end{aligned}
$$

Therefore $d * \kappa$ is

$$
\begin{aligned}
d * \kappa &= \epsilon_1\partial_1(\Phi\tilde{\Phi})dx^{13456} + \epsilon_1\partial_2(\Phi\tilde{\Phi})dx^{23456}\,,\\
&= \frac{\epsilon_1}{\Phi\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}}\left\{\partial_1(\Phi\tilde{\Phi})e^{13456} + \partial_2(\Phi\tilde{\Phi})e^{23456}\right\}\,,
\end{aligned}
$$

and thus we have

$$\begin{aligned}
d(e^{-2\varphi} * \kappa) &= e^{-2\varphi} d * \kappa - 2e^{-2\varphi} d\varphi \wedge *\kappa\,, \\
&= e^{-2\varphi} \frac{\epsilon_1}{\Phi\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \Big\{ \partial_1(\Phi\tilde{\Phi}) e^{13456} + \partial_2(\Phi\tilde{\Phi}) e^{23456} \Big\} \\
&\quad - e^{-2\varphi} \frac{1}{\Phi\tilde{\Phi}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}} \Big\{ \partial_1(\Phi\tilde{\Phi}) e^1 + \partial_2(\Phi\tilde{\Phi}) e^2 \Big\} \wedge (\epsilon_1 e^{3456} + \epsilon_2 e^{1256} + \epsilon_3 e^{1234})\,, \\
&= 0\,.
\end{aligned}$$

The rest task is to check (C.1). (3,0)-form Υ and fundamental 2-form κ have a volume matching condition [25],

$$\Upsilon \wedge \bar{\Upsilon} = i\frac{4}{3}\kappa^3\,. \tag{C.16}$$

Because Υ is (3,0)-form it takes the ensuing form under conditions $\epsilon_1 = -1, \epsilon_2 = 1, \epsilon_3 = 1$,

$$\Upsilon = h\zeta^{123}\,, \tag{C.17}$$

where ζ^1, ζ^2 and ζ^3 are given by

$$\zeta^1 = \sqrt{\phi\tilde{\phi}}(dx^1 + i dx^2)\,,\ \zeta^2 = \sqrt{\phi}dx^3 - i\frac{1}{\sqrt{\phi}}(dx^4 - \psi dx^3)\,,\ \zeta^3 = \sqrt{\tilde{\phi}}dx^5 - i\frac{1}{\sqrt{\tilde{\phi}}}(dx^6 - \tilde{\psi}dx^5). \tag{C.18}$$

Then, e^μ are rewritten as follows (see also Appendix 2):

$$\begin{aligned}
&e^1 = \frac{1}{2\sqrt{\phi\tilde{\phi}}}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}(\zeta^1 + \zeta^{\bar{1}}) = \frac{1}{2}\sqrt{\Phi\tilde{\Phi}}(\zeta^1 + \zeta^{\bar{1}})\,,\ e^2 = \frac{1}{2i\sqrt{\phi\tilde{\phi}}}\sqrt{\Phi\tilde{\Phi}\phi\tilde{\phi}}(\zeta^1 - \zeta^{\bar{1}}) = \frac{1}{2i}\sqrt{\Phi\tilde{\Phi}}(\zeta^1 - \zeta^{\bar{1}})\,, \\
&e^3 = \frac{1}{2\sqrt{\phi}}\sqrt{\Phi\phi}(\zeta^2 + \zeta^{\bar{2}}) = \frac{1}{2}\sqrt{\Phi}(\zeta^2 + \zeta^{\bar{2}})\,,\ e^4 = -\frac{\sqrt{\phi}}{2i}\sqrt{\frac{\Phi}{\phi}}(\zeta^2 - \zeta^{\bar{2}}) = -\frac{1}{2i}\sqrt{\Phi}(\zeta^2 - \zeta^{\bar{2}})\,, \\
&e^5 = \frac{1}{2\sqrt{\tilde{\phi}}}\sqrt{\tilde{\Phi}\tilde{\phi}}(\zeta^3 + \zeta^{\bar{3}}) = \frac{1}{2}\sqrt{\tilde{\Phi}}(\zeta^3 + \zeta^{\bar{3}})\,,\ e^6 = -\frac{\sqrt{\tilde{\phi}}}{2i}\sqrt{\frac{\tilde{\Phi}}{\tilde{\phi}}}(\zeta^3 - \zeta^{\bar{3}}) = -\frac{1}{2i}\sqrt{\tilde{\Phi}}(\zeta^3 - \zeta^{\bar{3}})\,.
\end{aligned}$$

Hence κ^3 is calculated as follows: e^{ab} $(a, b = 1, \cdots, 6)$ are written by means of ζ^i and $\zeta^{\bar{i}}$

$$\begin{aligned}
e^{12} &= -\frac{1}{2i}\Phi\tilde{\Phi}\zeta^1 \wedge \zeta^{\bar{1}} = \frac{i}{2}\Phi\tilde{\Phi}\zeta^1 \wedge \zeta^{\bar{1}}\,, \\
e^{34} &= \frac{1}{2i}\Phi\zeta^2 \wedge \zeta^{\bar{2}} = -\frac{i}{2}\Phi\zeta^2 \wedge \zeta^{\bar{2}}\,, \\
e^{56} &= \frac{1}{2i}\tilde{\Phi}\zeta^3 \wedge \zeta^{\bar{3}} = -\frac{i}{2}\tilde{\Phi}\zeta^3 \wedge \zeta^{\bar{3}}\,,
\end{aligned}$$

and thus κ^3 is given by

$$\begin{aligned}\kappa^3 &= 6\epsilon_1\epsilon_2\epsilon_3 e^{123456}\,, \\ &= \frac{3}{4}i\epsilon_1\epsilon_2\epsilon_3(\Phi\tilde{\Phi})^2\zeta^{123}\wedge\bar{\zeta}^{\bar{1}\bar{2}\bar{3}}\,.\end{aligned}$$

By (C.17) $\Upsilon\wedge\bar{\Upsilon}$ is also calculated as

$$\Upsilon\wedge\bar{\Upsilon} = h\bar{h}\zeta^{123}\wedge\bar{\zeta}^{\bar{1}\bar{2}\bar{3}}\,.$$

Therefore from (C.16) absolute value of the function h is given by

$$|h(z,\bar{z})|^2 = -\epsilon_1\epsilon_2\epsilon_3(\Phi\tilde{\Phi})^2 \tag{C.19}$$

and thus (3,0)-form Υ is written as

$$\Upsilon = \sqrt{-\epsilon_1\epsilon_2\epsilon_3(\Phi\tilde{\Phi})^2}\,f_p\zeta^{123}\,, \tag{C.20}$$

where f_p is a phase function and $|f_p| = 1$. Now, (3,0)-form Υ should satisfy (C.1) and we have

$$\begin{aligned}d(e^{-2\varphi}\Upsilon) &= d\left(\sqrt{\epsilon_1\epsilon_2\epsilon_3}f_p\zeta^{123}\right) \quad (\because e^{2\varphi} = \Phi\tilde{\Phi}) \\ &= \sqrt{\epsilon_1\epsilon_2\epsilon_3}(df_p\wedge\zeta^{123} + f_p d\zeta^1\wedge\zeta^{23} - f_p\zeta^1\wedge d\zeta^2\wedge\zeta^3 + f_p\zeta^{12}\wedge d\zeta^3) \\ &= \sqrt{\epsilon_1}\Bigg[(\partial_1 f_p)\frac{1}{2\sqrt{\phi\tilde{\phi}}}(\zeta^1+\bar{\zeta}^{\bar{1}}) + (\partial_2 f_p)\frac{1}{2i\sqrt{\phi\tilde{\phi}}}(\zeta^1-\bar{\zeta}^{\bar{1}}) + (\partial_3 f_p)\frac{1}{2\sqrt{\phi}}(\zeta^2+\bar{\zeta}^{\bar{2}}) \\ &\quad +(\partial_4 f_p)\left\{-\frac{\sqrt{\phi}}{2i}(\zeta^2-\bar{\zeta}^{\bar{2}}) + \frac{\psi}{2\sqrt{\phi}}(\zeta^2+\bar{\zeta}^{\bar{2}})\right\} + (\partial_5 f_p)\left\{\frac{1}{2\sqrt{\tilde{\phi}}}(\zeta^3+\bar{\zeta}^{\bar{3}})\right\} \\ &\quad +(\partial_6 f_p)\left\{-\frac{\sqrt{\tilde{\phi}}}{2i}(\zeta^3-\bar{\zeta}^{\bar{3}}) + \frac{\tilde{\psi}}{2\sqrt{\tilde{\phi}}}(\zeta^3+\bar{\zeta}^{\bar{3}})\right\}\Bigg]\zeta^{123} + 0 \\ &= \frac{\sqrt{\epsilon_1}}{2\sqrt{\phi\tilde{\phi}}}(\partial_1 f_p + i\partial_2 f_p)\bar{\zeta}^{\bar{1}}\zeta^{123} + \frac{\sqrt{\epsilon_1}}{2\sqrt{\phi}}\left[\partial_3 f_p + (\psi - i\phi)\partial_4 f_p\right]\bar{\zeta}^{\bar{2}}\zeta^{123} \\ &\quad + \frac{\sqrt{\epsilon_1}}{2\sqrt{\tilde{\phi}}}\left[\partial_5 f_p + (\tilde{\psi} - i\tilde{\phi})\partial_6 f_p\right]\bar{\zeta}^{\bar{3}}\zeta^{123}\end{aligned}$$

To satisfy (C.1), the following equation required for a phase function f_p:

$$\partial_1 f_p + i\partial_2 f_p = 0\,,\ \partial_3 f_p + (\psi - i\phi)\partial_4 f_p = 0\,,\ \partial_5 f_p + (\tilde{\psi} - i\tilde{\phi})\partial_6 f_p = 0\,. \tag{C.21}$$

Decomposing f_p into the real part and the imaginary part, $f_p = F_R + iF_I$, the equations about f_p are the following:

$$\begin{aligned}&\partial_1 F_R - \partial_2 F_I = 0\,,\ \partial_1 F_I + \partial_2 F_R = 0\,, \\ &\partial_3 F_R + \psi\partial_4 F_R + \phi\partial_4 F_I = 0\,,\ \partial_3 F_I - \phi\partial_4 F_R + \psi\partial_4 F_I = 0\,, \\ &\partial_5 F_R + \tilde{\psi}\partial_6 F_R + \tilde{\phi}\partial_6 F_I = 0\,,\ \partial_5 F_I - \tilde{\phi}\partial_6 F_R + \tilde{\psi}\partial_6 F_I = 0\,.\end{aligned} \tag{C.22}$$

Here $\phi, \psi, \tilde{\phi}, \tilde{\psi}$ depend only x_1, x_2 and thus we obtain the following relation by (C.21),

$$\frac{\partial_3 f_p}{\partial_4 f_p} = -(\psi - i\phi) \, \text{independent of } x_3, x_4 \ , \ \frac{\partial_5 f_p}{\partial_6 f_p} = -(\tilde{\psi} - i\tilde{\phi}) \, \text{independent of } x_5, x_6 \, . \tag{C.23}$$

Then, f_p is

$$f_p(x_1, x_2, x_3, x_4, x_5, x_6) = f(x_1, x_2) e^{i(c_3 x_3 + c_4 x_4)} e^{i(c_5 x_5 + c_6 x_6)} \, , \tag{C.24}$$

where c_3, c_4, c_5, c_6 are dependent on x_1 and x_2. Substituting this f_p for (C.21), the ensuing equations is lead;

$$c_3 + c_4(\psi - i\phi) = 0 \ , \ c_5 + c_6(\tilde{\psi} - i\tilde{\phi}) = 0 \, . \tag{C.25}$$

To be valid for any harmonic functions $\phi, \psi, \tilde{\phi}, \tilde{\psi}$, c_3, c_4, c_5, c_6 vanish,

$$c_3 = c_4 = c_5 = c_6 = 0 \, , \tag{C.26}$$

Therefore the phase function f_p is

$$f_p = f_p(x^1, x^2) \, . \tag{C.27}$$

Also f_p is holomorphic function because it's real part F_R and it's imaginary part F_I satisfy Cauchy–Riemann equations

$$\partial_1 F_R = \partial_2 F_I \ , \ \partial_1 F_I = -\partial_2 F_R \, . \tag{C.28}$$

Hence f_p is bounded on complex plane $\boldsymbol{C}$, an entire function. Let us recall Liouville's theorem,

theorem 1 (Liouville's theorem)**.** *If and only if an entire function is bounded, the function is a constant.*

By this theorem and $|f_p| = 1$,

$$f_p = 1 \, . \tag{C.29}$$

After all, Υ is written as

$$\Upsilon = \sqrt{\epsilon_1} \Phi \tilde{\Phi} \zeta^{123} \, , \quad (\because \epsilon_2 \epsilon_3 = 1 \text{ from } \epsilon_1 \epsilon_2 = -1 \text{ and } \epsilon_1 \epsilon_3 = -1) \tag{C.30}$$

or by using orthonormal base e^μ,

$$\Upsilon = \sqrt{\epsilon_1}(e^1 + i\, e^2) \wedge \sqrt{\epsilon_2}(e^3 - i\, e^4) \wedge \sqrt{\epsilon_3}(e^5 - i\, e^6) \, . \tag{C.31}$$

Clearly, the (3,0)-form Υ satisfy (C.1).

Consequently, the fundamental 2-form κ(4.24), the (3,0)-form Υ(4.31), the Lee form Θ(C.7), and the Bismut torsion T(4.26) satisfy (C.1), (C.2), and (C.3). Namely, the triplet (g, T, φ),

$$g_6 = \Phi\tilde{\Phi}\phi\tilde{\phi}\{(dx^1)^2 + (dx^2)^2\} + \Phi\phi(dx^3)^2 + \frac{\Phi}{\phi}(dx^4 - \psi dx^3)^2 + \tilde{\Phi}\tilde{\phi}(dx^5)^2 + \frac{\tilde{\Phi}}{\tilde{\phi}}(dx^6 - \tilde{\psi} dx^5)^2 \, , \tag{C.32}$$

$$T = -(\partial_2 \Phi dx^{134} - \partial_1 \Phi dx^{234}) - (\partial_2 \tilde{\Phi} dx^{156} - \partial_1 \tilde{\Phi} dx^{256}) \, , \tag{C.33}$$

$$\varphi = \log \sqrt{\Phi\tilde{\Phi}} + const. \ , \tag{C.34}$$

constructs the geometry admitting the $SU(3)$-structure classified by (C.1), (C.2) and (C.3). Obviously, this result is valid for the condition $\phi = \tilde{\phi} = \Phi = \tilde{\Phi}$.

C.1 Curvature

When $\Phi = \tilde{\Phi} = \phi = \tilde{\phi}$, the components of the Bismut curvature are listed as follows:

$$R^{+}_{1212} = -\frac{2}{\phi^4}(\partial_1{}^2 \log|\phi| + \partial_2{}^2 \log|\phi|)\,,$$

$$R^{+}_{3412} = R^{+}_{5612} = -\frac{1}{\phi^4}(\partial_1{}^2 \log|\phi| + \partial_2{}^2 \log|\phi|)\,,$$

$$R^{+}_{1313} = -R^{+}_{2413} = -\frac{1}{\phi^4}\left(\partial_1^2 \log|\phi| - (\partial_1 \log|\phi|)^2 + 3(\partial_2 \log|\phi|)^2\right)\,,$$

$$R^{+}_{1323} = -R^{+}_{2423} = -\frac{1}{\phi^4}\left(\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\right)\,,$$

$$R^{+}_{1413} = R^{+}_{2313} = -\frac{1}{\phi^4}(\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|))\,,$$

$$R^{+}_{1423} = R^{+}_{2323} = -\frac{1}{\phi^4}\left(\partial_2{}^2 \log|\phi| - (\partial_2 \log|\phi|)^2 + 3(\partial_1 \log|\phi|)^2\right)\,,$$

$$R^{+}_{1515} = -R^{+}_{2615} = -\frac{1}{\phi^4}\left(\partial_1{}^2 \log|\phi| - (\partial_1 \log|\phi|)^2 + 3(\partial_2 \log|\phi|)^2\right)\,,$$

$$R^{+}_{1525} = -R^{+}_{2625} = -\frac{1}{\phi^4}\left(\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\right)\,,$$

$$R^{+}_{1615} = R^{+}_{2515} = -\frac{1}{\phi^4}(\,\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^{+}_{1625} = R^{+}_{2525} = -\frac{1}{\phi^4}(\,\partial_2{}^2 \log|\phi| - (\partial_2 \log|\phi|)^2 + 3(\partial_1 \log|\phi|)^2\,)\,,$$

$$R^{+}_{3535} = R^{+}_{4635} = -\frac{1}{\phi^4}(\,(\partial_1 \log|\phi|)^2 + (\partial_2 \log|\phi|)^2\,)\,,$$

$$\begin{aligned}
&R^{+}_{1234} = R^{+}_{1256} = R^{+}_{1314} = R^{+}_{1324} = R^{+}_{1414} = R^{+}_{1424} = R^{+}_{1516} = R^{+}_{1526} = R^{+}_{1616} = R^{+}_{1626} = 0\,,\\
&R^{+}_{2314} = R^{+}_{2324} = R^{+}_{2414} = R^{+}_{2424} = R^{+}_{2516} = R^{+}_{2526} = R^{+}_{2616} = R^{+}_{2626} = 0\,,\\
&R^{+}_{3434} = R^{+}_{3536} = R^{+}_{3545} = R^{+}_{3546} = R^{+}_{3635} = R^{+}_{3645} = R^{+}_{4535} = R^{+}_{4536} = R^{+}_{5656} = 0\,.
\end{aligned}$$

Also, the components of the Hull curvature are listed as follows:

$$R^{-}_{1212} = -\frac{2}{\phi^4}(\partial_1{}^2 \log|\phi| + \partial_2{}^2 \log|\phi|)\,,\; R^{-}_{1234} = R^{-}_{1256} = \frac{1}{\phi^4}(\,(\partial_1 \log|\phi|)^2 + (\partial_2 \log|\phi|)^2\,)\,,$$

$$R^{-}_{1313} = -\frac{1}{\phi^4}(\,\partial_1^2 \log|\phi| - (\partial_1 \log|\phi|)^2 + 3(\partial_2 \log|\phi|)^2\,)\,,$$

$$R^{-}_{1314} = -\frac{1}{\phi^4}(\,\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{1323} = -\frac{1}{\phi^4}(\,\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{1324} = -\frac{1}{\phi^4}(\,{\partial_2}^2 \log|\phi| - 2(\partial_2 \log|\phi|)^2 + 2(\partial_1 \log|\phi|)^2\,)\,,$$

$$R^-_{1515} = -\frac{1}{\phi^4}(\,{\partial_1}^2 \log|\phi| - (\partial_1 \log|\phi|)^2 + 3(\partial_2 \log|\phi|)^2\,)\,,$$

$$R^-_{1516} = -\frac{1}{\phi^4}(\,\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{1525} = -\frac{1}{\phi^4}(\,\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{1526} = -\frac{1}{\phi^4}(\,({\partial_2}^2 \log|\phi|) - 2(\partial_2 \log|\phi|)^2 + 2(\partial_1 \log|\phi|)^2\,)\,,$$

$$R^-_{2313} = \frac{1}{\phi^4}(\,-\partial_1\partial_2 \log|\phi| + 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{2314} = \frac{1}{\phi^4}(\,{\partial_1}^2 \log|\phi| - 2(\partial_1 \log|\phi|)^2 + 2(\partial_2 \log|\phi|)^2\,)\,,$$

$$R^-_{2323} = \frac{1}{\phi^4}(\,-{\partial_2}^2 \log|\phi| + (\partial_2 \log|\phi|)^2 - 3(\partial_1 \log|\phi|)^2\,)\,,$$

$$R^-_{2324} = \frac{1}{\phi^4}(\,\partial_2\partial_1 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{2515} = \frac{1}{\phi^4}(\,-\partial_1\partial_2 \log|\phi| + 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{2516} = \frac{1}{\phi^4}(\,{\partial_1}^2 \log|\phi| - 2(\partial_1 \log|\phi|)^2 + 2(\partial_2 \log|\phi|)^2\,)\,,$$

$$R^-_{2525} = \frac{1}{\phi^4}(\,-{\partial_2}^2 \log|\phi| + (\partial_2 \log|\phi|)^2 - 3(\partial_1 \log|\phi|)^2\,)\,,$$

$$R^-_{2526} = \frac{1}{\phi^4}(\,\partial_1\partial_2 \log|\phi| - 4(\partial_1 \log|\phi|)(\partial_2 \log|\phi|)\,)\,,$$

$$R^-_{3535} = -\frac{1}{\phi^4}(\,(\partial_1 \log|\phi|)^2 + (\partial_2 \log|\phi|)^2\,)\,,$$

$$R^-_{3546} = -\frac{1}{\phi^4}(\,(\partial_2 \log|\phi|)^2 + (\partial_1 \log|\phi|)^2\,)\,,$$

$$\begin{aligned}
&R^-_{1413} = R^-_{1414} = R^-_{1423} = R^-_{1424} = R^-_{1615} = R^-_{1616} = R^-_{1625} = R^-_{1626} = 0\,,\\
&R^-_{2413} = R^-_{2414} = R^-_{2423} = R^-_{2424} = R^-_{2615} = R^-_{2616} = R^-_{2625} = R^-_{2626} = 0\,,\\
&R^-_{3412} = R^-_{3434} = R^-_{3536} = R^-_{3545} = R^-_{3635} = R^-_{3645} = R^-_{4535} = R^-_{4536} = R^-_{4635} = R^-_{5612} = R^-_{5656} = 0\,.
\end{aligned}$$

Appendix D

$SU(3)$ structure of the T-dualized solution along ∂_3

We T-dualize the solutions (4.47), (4.48), and (4.49) along ∂_3 and then they take the following form:

$$\begin{aligned} \hat{g}_6 &= \frac{1}{(\phi^2+\psi^2)}(d\hat{x}^3 - B_1 dx^4)^2 + \phi^4\{(dx^1)^2 + (dx^2)^2\} + \frac{\phi^2}{\phi^2+\psi^2}(dx^4)^2 && \text{(D.1)} \\ &\quad +\phi^2(dx^5)^2 + (dx^6 - \psi dx^5)^2\,, && \text{(D.2)} \\ \hat{B} &= -\frac{\psi}{\phi^2+\psi^2}dx^4 \wedge d\hat{x}^3 + B_2 dx^{56}\,, && \text{(D.3)} \\ e^{2\hat{\varphi}} &= \frac{1}{(\phi^2+\psi^2)}e^{2\varphi}\,. && \text{(D.4)} \end{aligned}$$

Let us introduce an new orthonormal frame;

$$\begin{aligned} &\hat{e}^1 = \phi^2 dx^1\,,\ \hat{e}^2 = \phi^2 dx^2\,,\ \hat{e}^3 = \frac{1}{\sqrt{\phi^2+\psi^2}}(d\hat{x}^3 - B_1 dx^4)\,,\ \hat{e}^4 = \frac{\phi}{\sqrt{\phi^2+\psi^2}}dx^4\,, \\ &\hat{e}^5 = \phi dx^5\,,\ \hat{e}^6 = dx^6 - \psi dx^5 && \text{(D.5)} \end{aligned}$$

and exterior derivatives of them are given by

$$\begin{aligned} d\hat{e}^1 &= -2\phi\frac{\partial\phi}{\partial x^2}dx^{12} = -\frac{2}{\phi^3}\frac{\partial\phi}{\partial x^2}\hat{e}^{12}\,,\ d\hat{e}^2 = \frac{2}{\phi^3}\frac{\partial\phi}{\partial x^1}\hat{e}^{12}\,, \\ d\hat{e}^3 &= \frac{1}{\phi^2(\phi^2+\psi^2)}(\,-(\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^{13} - (\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^{23}\,) - \frac{1}{\phi^3}\partial_1 B_1\hat{e}^{14} - \frac{1}{\phi^3}\partial_2 B_1 \hat{e}^{24}\,, \\ d\hat{e}^4 &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\psi\partial_1\phi + \phi\partial_2\phi)\hat{e}^{14} + (\psi\partial_2\phi - \phi\partial_1\phi)\hat{e}^{24}\,)\,, \\ d\hat{e}^5 &= \frac{1}{\phi^3}(\partial_1\phi\hat{e}^{15} + \partial_2\phi\hat{e}^{25})\,, \\ d\hat{e}^6 &= \frac{1}{\phi^3}(\partial_2\phi\hat{e}^{15} - \partial_1\phi\hat{e}^{25})\,. && \text{(D.6)} \end{aligned}$$

Thus we can read the structure constants $\hat{f}^{\mu}_{\nu\rho}$ which are defined by $d\hat{e}^{\mu} = \frac{1}{2}\hat{f}^{\mu}_{\nu\rho}\hat{e}^{\nu\rho}$ as follows;

$$\begin{aligned}
&\hat{f}^1{}_{12} = -\frac{2}{\phi^3}\partial_2\phi\,,\ \hat{f}^2{}_{12} = \frac{2}{\phi^3}\partial_1\phi\,,\\
&\hat{f}^3{}_{13} = -\frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_1\phi - \psi\partial_2\phi)\,,\ \hat{f}^3{}_{23} = -\frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_2\phi + \psi\partial_1\phi)\,,\\
&\hat{f}^3{}_{14} = -\frac{1}{\phi^3}\partial_1 B_1\,,\ \hat{f}^3{}_{24} = -\frac{1}{\phi^3}\partial_2 B_1\,,\\
&\hat{f}^4{}_{14} = \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\psi\partial_1\phi + \phi\partial_2\phi)\,,\ \hat{f}^4{}_{24} = \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\psi\partial_2\phi - \phi\partial_1\phi)\,,\\
&\hat{f}^5{}_{15} = \frac{1}{\phi^3}\partial_1\phi\,,\ \hat{f}^5{}_{25} = \frac{1}{\phi^3}\partial_2\phi\,,\ \hat{f}^6{}_{15} = \frac{1}{\phi^3}\partial_2\phi\,,\ \hat{f}^6{}_{25} = -\frac{1}{\phi^3}\partial_1\phi\,.
\end{aligned} \tag{D.7}$$

A flux of the B-field is defined by

$$\hat{H} = -d\hat{B} \tag{D.8}$$

and thus the flux $\hat{H}$ is given by

$$\begin{aligned}
\hat{H} \;=\; & -\frac{1}{\phi^3(\phi^2+\psi^2)}\big(\,((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)\hat{e}^{134} + ((\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi)\hat{e}^{234}\,\big)\\
& -\frac{1}{\phi^3}(\partial_1 B_2)\hat{e}^{156} - \frac{1}{\phi^3}(\partial_2 B_2)\hat{e}^{256}\,.
\end{aligned} \tag{D.9}$$

We should investigate that the triplet $(\hat{g}_6, \hat{H}, \hat{\varphi})$ satisfies supersymmetry preserving conditions:

$$d(e^{-2\hat{\varphi}} * \hat{\kappa}) = 0\,, \tag{D.10}$$

$$d(e^{-2\hat{\varphi}}\hat{\Upsilon}) = 0\,, \tag{D.11}$$

$$*\hat{H} = e^{2\hat{\varphi}} d(e^{-2\hat{\varphi}}\hat{\kappa})\,, \tag{D.12}$$

where $\hat{J}$, $\hat{\kappa}$, $\hat{\Upsilon}^{(3,0)}$ represent an almost complex structure, a fundamental 2-form, and a (3,0)-form respectively.

We assume that a fundamental 2-form takes the form

$$\hat{\kappa} = K(x^1, x^2)(\varepsilon_1\hat{e}^{12} + \varepsilon_2\hat{e}^{34} + \varepsilon_3\hat{e}^{56})\,. \tag{D.13}$$

Let us determine a function $K(x^1, x^2)$ such that $\hat{\kappa}$ satisfies (D.10). We define a volume form to the metric (D.2) by

$$\text{vol.} = *1\ ,\ \text{vol.} = \hat{e}^{123456}\,. \tag{D.14}$$

Then Hodge dual of $\hat{e}^{12}$, $\hat{e}^{34}$ and $\hat{e}^{56}$ are given by

$$*\hat{e}^{12} = \iota_{\hat{e}_2}\iota_{\hat{e}_1}\text{vol.} = \hat{e}^{3456}\ ,\ *\hat{e}^{34} = \hat{e}^{1256}\ ,\ *\hat{e}^{56} = \hat{e}^{1234}\,.$$

We have

$$*\hat{\kappa} = K(\varepsilon_1\hat{e}^{3456} + \varepsilon_2\hat{e}^{1256} + \varepsilon_3\hat{e}^{1234}) \tag{D.15}$$

and

$$e^{-2\hat{\varphi}} * \hat{\kappa} = \frac{Kh}{\phi^2}(\varepsilon_1 \hat{e}^{3456} + \varepsilon_2 \hat{e}^{1256} + \varepsilon_3 \hat{e}^{1234}) ,$$

where

$$h \equiv \phi^2 + \psi^2 . \tag{D.16}$$

Therefore the left hand side of (D.10) takes the form,

$$\begin{aligned} d(e^{-2\hat{\varphi}} * \hat{\kappa}) &= \frac{K}{\phi^2} dh \wedge (\varepsilon_1 \hat{e}^{3456} + \varepsilon_2 \hat{e}^{1256} + \varepsilon_3 \hat{e}^{1234}) - \frac{hK}{\phi^4} d(\phi\tilde{\phi}) \wedge (\varepsilon_1 \hat{e}^{3456} + \varepsilon_2 \hat{e}^{1256} + \varepsilon_3 \hat{e}^{1234}) \\ &\quad + \frac{h}{\phi^2} dK \wedge (\varepsilon_1 \hat{e}^{3456} + \varepsilon_2 \hat{e}^{1256} + \varepsilon_3 \hat{e}^{1234}) + \frac{hK}{\phi^2}(\varepsilon_1 d\hat{e}^{3456} + \varepsilon_2 d\hat{e}^{1256} + \varepsilon_3 d\hat{e}^{1234}) \end{aligned} \tag{D.17}$$

and each terms of (D.17) are the following:

$$1st\, term = \varepsilon_1 \frac{2K\phi^2}{\phi^6}(\, (\phi\partial_1\phi - \psi\partial_2\phi)\hat{e}^{13456} + (\phi\partial_2\phi + \psi\partial_1\phi)\hat{e}^{23456}\,) ,$$

$$2nd\, term = -\varepsilon_1 \frac{hK}{\phi^6}(\, \partial_1\phi^2 \hat{e}^{13456} + \partial_2\phi^2 \hat{e}^{23456}\,) ,$$

$$3rd\, term = \varepsilon_1 \frac{h}{\phi^6}(\, \phi^2(\partial_1 K)\hat{e}^{13456} + \phi^2(\partial_2 K)\hat{e}^{23456}\,) ,$$

$$4th\, term = \varepsilon_1 \frac{hK}{\phi^6}(\, \phi^4(\hat{f}^3{}_{13} + \hat{f}^4{}_{14} + \hat{f}^5{}_{15})\hat{e}^{13456} + \phi^4(\hat{f}^3{}_{23} + \hat{f}^4{}_{24} + \hat{f}^5{}_{25})\hat{e}^{23456}\,) .$$

Then we have

$$\begin{aligned} d(e^{-2\hat{\varphi}} * \hat{\kappa}) &= \frac{\varepsilon_1}{\phi^6}(\, h(\phi^2\partial_1 K + \phi^4 K(\hat{f}^3{}_{13} + \hat{f}^4{}_{14} + \hat{f}^5{}_{15})) - hK\partial_1\phi^2 + 2K\phi^2(\phi\partial_1\phi - \psi\partial_2\phi)\,)\hat{e}^{13456} \\ &\quad + \frac{\varepsilon_1}{\phi^6}(\, h(\phi^2\partial_2 K + \phi^4 K(\hat{f}^3{}_{23} + \hat{f}^4{}_{24} + \hat{f}^5{}_{25})) - hK\partial_2\phi^2 + 2K\phi^2(\phi\partial_2\phi + \psi\partial_1\phi)\,)\hat{e}^{23456} , \end{aligned}$$

where

$$h\phi^4(\hat{f}^3{}_{13} + \hat{f}^4{}_{14} + \hat{f}^5{}_{15}) \;=\; -\phi^2(\phi\partial_1\phi - \psi\partial_2\phi) + \psi^2\phi\partial_1\phi + \phi^2\psi\partial_2\phi + h\phi\partial_1\phi$$

and

$$h\phi^4(\hat{f}^3{}_{23} + \hat{f}^4{}_{24} + \hat{f}^5{}_{25}) \;=\; -\phi^2(\phi\partial_2\phi + \psi\partial_1\phi) + \psi^2\phi\partial_2\phi - \phi^2\psi\partial_1\phi + h\phi\partial_2\phi .$$

Therefore we have

$$d(e^{-2\hat{\varphi}} * \hat{\kappa}) \;=\; \frac{\varepsilon_1}{\phi^6} h\phi^2(\, (\partial_1 K)\hat{e}^{13456} + (\partial_2 K)\hat{e}^{23456}\,)$$

and thus we obtain the following equations for the function $K(x^1, x^2)$ by (D.10)

$$\partial_1 K = 0\;,\;\; \partial_2 K = 0 .$$

Then, the function $K(x^1, x^2) = 1$. Consequently, the fundamental 2-form $\hat{\kappa}$ takes the form

$$\hat{\kappa} = \varepsilon_1 \hat{e}^{12} + \varepsilon_2 \hat{e}^{34} + \varepsilon_3 \hat{e}^{56} \,. \tag{D.18}$$

From (D.18), an almost complex structure $\hat{J}$ is defined by

$$\hat{J}\hat{e}_1 = -\varepsilon_1 \hat{e}_2 \;,\; \hat{J}\hat{e}_3 = -\varepsilon_2 \hat{e}_4 \;,\; \hat{J}\hat{e}_5 = -\varepsilon_3 \hat{e}_6 \tag{D.19}$$

and $\hat{J}_* \hat{e}^\mu$ $(\mu = 1, \cdots 6)$ are given by

$$\hat{J}_*\hat{e}_1 = \varepsilon_1 \hat{e}_2 \;,\; \hat{J}_*\hat{e}_3 = \varepsilon_2 \hat{e}_4 \;,\; \hat{J}_*\hat{e}_5 = \varepsilon_3 \hat{e}_6 \,. \tag{D.20}$$

A Nijenhuis tensor N associated with the almost complex structure $\hat{J}$ is defined by

$$N_{\hat{J}}(X, Y) = [\hat{J}X \,,\, \hat{J}Y] - [X \,,\, Y] - \hat{J}[\hat{J}X \,,\, Y] - \hat{J}[X \,,\, \hat{J}Y] \,, \tag{D.21}$$

where a commutation relation between vector fields are given by $[\hat{e}_\mu \,,\, \hat{e}_\nu] = -\hat{f}^\rho{}_{\mu\nu}\hat{e}_\rho$. We require $\varepsilon_1\varepsilon_2 = -1$ and then we have

$$\begin{aligned}
N_{\hat{J}}(\hat{e}_1, \hat{e}_3) &= \frac{1}{\phi^3}(\, -(\partial_1\phi + \partial_2 B_1)\hat{e}_3 + (\partial_2\phi - \partial_1 B_1)\hat{e}_4 \,) \,, \\
N_{\hat{J}}(\hat{e}_2, \hat{e}_3) &= \frac{1}{\phi^3}(\, (-\partial_2\phi + \partial_1 B_1)\hat{e}_3 - (\partial_1\phi + \partial_2 B_1)\hat{e}_4 \,) \,, \\
N_{\hat{J}}(\hat{e}_1, \hat{e}_4) &= \frac{1}{\phi^3}(\partial_2\phi - \partial_1 B_1)\hat{e}_3 + \frac{1}{\phi^3}(\partial_1\phi + \partial_2 B_1)\hat{e}_4 \,, \\
N_{\hat{J}}(\hat{e}_2, \hat{e}_4) &= -\frac{1}{\phi^3}(\partial_1\phi + \partial_2 B_1)\hat{e}_3 - \frac{1}{\phi^3}(-\partial_2\phi + \partial_1 B_1)\hat{e}_4 \,.
\end{aligned} \tag{D.22}$$

If $B_1 = -\psi$, then from Cauchy–Riemann equations (4.20) the components of the Nijenhuis tensor are the following,

$$N_{\hat{J}}(\hat{e}_1, \hat{e}_3) = 0 \;,\; N_{\hat{J}}(\hat{e}_2, \hat{e}_3) = 0 \;,\; N_{\hat{J}}(\hat{e}_1, \hat{e}_4) = 0 \;,\; N_{\hat{J}}(\hat{e}_2, \hat{e}_4) = 0 \,. \tag{D.23}$$

Other components of the Nijenhuis tensor $N_{\hat{J}}(\hat{e}_\mu, \hat{e}_\nu)$ are also clearly zero under the condition $\varepsilon_1\varepsilon_3 = -1$. Hence the almost complex structure $\hat{J}$ is a complex structure when $B_1 = -\psi$, $\varepsilon_1\varepsilon_2 = \varepsilon_1\varepsilon_3 = -1$.

A Bismut torsion associated with $\hat{J}$ is defined by

$$\hat{T} := \varepsilon_4 \hat{J}_* d\hat{\kappa} \,. \tag{D.24}$$

The exterior derivative of $\hat{\kappa}$ is given by

$$\begin{aligned}
d\hat{\kappa} \;=\; & \frac{\varepsilon_2}{h\phi^3}(\, (\psi^2 - \phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi \,)\hat{e}^{134} + \frac{\varepsilon_2}{h\phi^3}(\, (\psi^2 - \phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi \,)\hat{e}^{234} \\
& + \frac{\varepsilon_3}{\phi^3}\partial_1\phi\hat{e}^{156} + \frac{\varepsilon_3}{\phi^3}\partial_2\phi\hat{e}^{256}
\end{aligned} \tag{D.25}$$

and we have

$$\begin{aligned}
\hat{J}_* d\hat{\kappa} &= \frac{\varepsilon_2}{h\phi^3}(\,(\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi\,)\hat{J}_*\hat{e}^1\wedge\hat{J}_*\hat{e}^3\wedge\hat{J}_*\hat{e}^4 \\
&\quad + \frac{\varepsilon_2}{h\phi^3}(\,(\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi\,)\hat{J}_*\hat{e}^2\wedge\hat{J}_*\hat{e}^3\wedge\hat{J}_*\hat{e}^4 \\
&\quad + \frac{\varepsilon_3}{\phi^3}\partial_1\tilde{\phi}\hat{J}_*\hat{e}^1\wedge\hat{J}_*\hat{e}^5\wedge\hat{J}_*\hat{e}^6 + \frac{\varepsilon_3}{\phi^3}\partial_2\phi\hat{J}\hat{e}^2\wedge\hat{J}\hat{e}^5\wedge\hat{J}\hat{e}^6 \\
&= -\frac{1}{h\phi^3}(\,(\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi\,)\hat{e}^{234} + \frac{1}{h\phi^3}(\,(\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi\,)\hat{e}^{134} \\
&\quad - \frac{1}{\phi^3}\partial_1\phi\hat{e}^{256} + \frac{1}{\phi^3}\partial_2\phi\hat{e}^{156}\,.
\end{aligned}$$

Hence the Bismut torsion takes the following form,

$$\hat{T} = \frac{\varepsilon_4}{h\phi^3}(\,(\,(\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi\,)\hat{e}^{234} + (\,(\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi\,)\hat{e}^{134} - h(\partial_1\phi\hat{e}^{256} - \partial_2\phi\hat{e}^{156})\,)\,.$$

On the other hand, deformed flux $\hat{H}$ with the conditions $B_1 = -\psi$ and $B_2 = -\psi$ is given by

$$\hat{H} = \frac{-1}{h\phi^3}(\,(\,(\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi\,)\hat{e}^{134} + (\,(\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi\,)\hat{e}^{234} + h(\partial_2\phi\hat{e}^{156} - \partial_1\phi\hat{e}^{256})\,)$$

and thus we see that $\hat{H} = \hat{T}$ when $\varepsilon_4 = -1$.

Let us check whether the torsion $\hat{T}$ with $\varepsilon_4 = -1$ satisfies the equation $d^{\hat{T}}\kappa = 0$ or not. According to Ref. [57], an exterior derivative with a torsion for deformed fundamental 2-form $\hat{\kappa}$ is given by

$$d^{\hat{T}}\hat{\kappa} = d\hat{\kappa} - \sum_{\mu=1}^{6}(\iota_{\hat{e}_\mu}\hat{T})\wedge(\iota_{\hat{e}_\mu}\hat{\kappa})\,. \tag{D.26}$$

The first term of $d^{\hat{T}}\hat{\kappa}$ has been given by (D.25). On the second term, $(\iota_{\hat{e}_\mu}\hat{T})\wedge(\iota_{\hat{e}_\mu}\hat{\kappa})$ for each values of $\mu = 1,\cdots,6$ are given by

$$(\iota_{\hat{e}_1}\hat{T})\wedge(\iota_{\hat{e}_1}\hat{\kappa}) = \frac{\varepsilon_1\varepsilon_4}{h\phi^3}(\,(\,(\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi\,)\hat{e}^{234} + h\partial_2\phi\hat{e}^{256}\,)\,,$$

$$(\iota_{\hat{e}_2}\hat{T})\wedge(\iota_{\hat{e}_2}\hat{\kappa}) = \frac{\varepsilon_1\varepsilon_4}{h\phi^3}(\,(\,(\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi\,)\hat{e}^{134} + h\partial_1\phi\hat{e}^{156}\,)\,,$$

$$(\iota_{\hat{e}_3}\hat{T})\wedge(\iota_{\hat{e}_3}\hat{\kappa}) = 0\,,\ (\iota_{\hat{e}_4}\hat{T})\wedge(\iota_{\hat{e}_4}\hat{\kappa}) = 0\,,\ (\iota_{\hat{e}_5}\hat{T})\wedge(\iota_{\hat{e}_5}\hat{\kappa}) = 0\,,\ (\iota_{\hat{e}_6}\hat{T})\wedge(\iota_{\hat{e}_6}\hat{\kappa}) = 0$$

and thus we have

$$\begin{aligned}
\sum_{\mu=1}^{6}(\iota_{\hat{e}_\mu}\hat{T})\wedge(\iota_{\hat{e}_\mu}\hat{\kappa}) &= \frac{\varepsilon_1\varepsilon_4}{h\phi^3}\big(\,(\,(\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi\,)\hat{e}^{134} \\
&\quad + (\,(\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi\,)\hat{e}^{234} + h(\partial_1\phi\hat{e}^{156} + \partial_2\phi\hat{e}^{256})\big)\,.
\end{aligned}$$

Therefore $d^{\hat{T}}\hat{\kappa}$ is

$$\begin{aligned} d^{\hat{T}}\hat{\kappa} &= \frac{\varepsilon_2}{h\phi^3}(\,(\,(\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi\,)\hat{e}^{134}+(\,(\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi\,)\hat{e}^{234}\,) \\ &+\frac{\varepsilon_3}{\phi^3}(\partial_1\phi\hat{e}^{156}+\partial_2\phi\hat{e}^{256})-\frac{\varepsilon_1\varepsilon_4}{h\phi^3}\left((\,(\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi\,)\hat{e}^{134}\right. \\ &\left.+(\,(\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi\,)\hat{e}^{234}+h(\partial_1\phi\hat{e}^{156}+\partial_2\phi\hat{e}^{256})\right) \end{aligned}$$

and thus we see that $d^{\hat{T}}\hat{\kappa}=0$ when $\varepsilon_1\varepsilon_2=-1$, $\varepsilon_1\varepsilon_3=-1$ and $\varepsilon_4=-1$.

A Lee form $\hat{\Theta}$ is defined by

$$\hat{\Theta}=\varepsilon_4\hat{J}_*\delta\hat{\kappa}\,, \tag{D.27}$$

where the operator δ is a co-derivative on p-form that is defined by $\delta=(-1)^{np-p^2+p}*d*$. The exterior derivatives of $*\hat{\kappa}$ is given by

$$d*\hat{\kappa}=\frac{2\varepsilon_1\psi}{\phi^3(\phi^2+\psi^2)}(\,(\psi\partial_1\phi+\phi\partial_2\phi)\hat{e}^{13456}+(-\phi\partial_1\phi+\psi\partial_2\phi)\hat{e}^{23456}\,) \tag{D.28}$$

and thus the Lee form takes the following form

$$\hat{\Theta}=-\varepsilon_4 d\left(\log\frac{\phi^2}{\phi^2+\psi^2}\right)\,. \tag{D.29}$$

If we identify a dilaton with the Lee form by $d\hat{\varphi}=-\frac{1}{2}\varepsilon_4\hat{\Theta}$, the dilaton $\hat{\varphi}$ is given by

$$\hat{\varphi}=\frac{1}{2}\log\frac{\phi^2}{\phi^2+\psi^2} \tag{D.30}$$

and this is consistent with (4.52).

We consider (D.12). The Hodge dual of $\hat{H}$ is given by

$$\begin{aligned} *\hat{H} &= -\frac{1}{\phi^3(\phi^2+\psi^2)}(\,(\,(\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi\,)\hat{e}^{256}-(\,(\phi^2-\psi^2)\partial_1\phi-2\phi\psi\partial_2\phi\,)\hat{e}^{156}\,) \\ &-\frac{1}{\phi^3}\partial_1 B_2\hat{e}^{234}+\frac{1}{\phi^3}\partial_2 B_2\hat{e}^{134}\,. \end{aligned} \tag{D.31}$$

The right hand side of (D.12) becomes

$$\begin{aligned} e^{2\hat{\varphi}}d(e^{-2\hat{\varphi}}\hat{\kappa}) &= -\frac{\varepsilon_2}{h\phi^3}\partial_1\phi\hat{e}^{134}-\frac{\varepsilon_2}{h\phi^3}\partial_2\phi\hat{e}^{234}+\frac{\varepsilon_3}{h\phi^3}(\,(\phi^2-\psi^2)\partial_1\phi-2\phi\psi\partial_2\phi\,)\hat{e}^{156} \\ &+\frac{\varepsilon_3}{h\phi^3}(\,(\phi^2-\psi^2)\partial_2\phi+2\phi\psi\partial_1\phi\,)\hat{e}^{256} \end{aligned}$$

and thus we have

$$\begin{aligned} *\hat{H}-e^{2\hat{\varphi}}d(e^{-2\hat{\varphi}}\hat{\kappa}) &= \frac{1}{\phi^3}(\partial_2 B_2+\varepsilon_2\partial_1\phi)\hat{e}^{134}-\frac{1}{\phi^3}(\partial_1 B_2-\varepsilon_2\partial_2\phi)\hat{e}^{234} \\ &+\frac{1}{h\phi^3}(\,(\phi^2-\psi^2)\partial_1\phi-2\phi\psi\partial_2\phi-\varepsilon_3(\,(\phi^2-\psi^2)\partial_1\phi-2\phi\psi\partial_2\phi\,)\,)\hat{e}^{156} \\ &-\frac{1}{h\phi^3}(\,(\psi^2-\phi^2)\partial_2\phi-2\phi\phi\partial_1\phi+\varepsilon_3(\,(\phi^2-\psi^2)\partial_2\phi+2\phi\psi\partial_1\phi\,)\,)\hat{e}^{256}\,. \end{aligned}$$

If $\varepsilon_2 = \varepsilon_3 = 1$ and $B_1 = B_2 = -\psi$, we have $*\hat{H} = e^{2\hat{\varphi}} d(e^{-2\hat{\varphi}}\hat{\kappa})$. Namely, the triplet $(\hat{g}, \hat{H}, \hat{\varphi})$ satisfies (D.12) when $\varepsilon_2 = \varepsilon_3 = 1$ and $B_1 = B_2 = -\psi$.

We consider a (3,0)-form $\hat{\Upsilon}$ which satisfies (D.11).We assume that $\hat{\Upsilon}$ takes the form

$$\begin{aligned}
\hat{\Upsilon} &= \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\hat{W}(x^1, x^2)(\hat{e}^1 + i\,\hat{e}^2) \wedge (\hat{e}^3 - i\,\hat{e}^4) \wedge (\hat{e}^5 - i\,\hat{e}^6) \\
&= \sqrt{\varepsilon_1\varepsilon_2\varepsilon_3}\hat{W}(x^1, x^2)(\,\hat{e}^{135} + \hat{e}^{245} + \hat{e}^{236} - \hat{e}^{146} + i(\hat{e}^{235} - \hat{e}^{145} - \hat{e}^{136} - \hat{e}^{246})\,)\,, \qquad \text{(D.32)}
\end{aligned}$$

where the function $\hat{W}$ is a complex function. A $(3,0)$-form $\hat{\Upsilon}$ has the relation between the fundamental 2-form $\hat{\kappa}$ by the volume matching condition,

$$\hat{\Upsilon} \wedge \bar{\hat{\Upsilon}} = \frac{4}{3} i\,\hat{\kappa}^3\,. \qquad \text{(D.33)}$$

Calculating (D.33) directly, we obtain the conditions

$$\varepsilon_1\varepsilon_2\varepsilon_3 = -1\,,\ |\hat{W}|^2 = 1 \qquad \text{(D.34)}$$

and thus the complex function $\hat{W}$ is a phase function,

$$\hat{W}(x^1, x^2) = f_p(x^1, x^2)\,, \qquad \text{(D.35)}$$

where $|f_p| = 1$. We determine f_p so as to satisfy (D.11). $e^{-2\hat{\varphi}}\hat{\Upsilon}$ is given by

$$e^{-2\hat{\varphi}}\hat{\Upsilon} = i\,\frac{hf_p}{\phi^2}(\,\hat{e}^{135} + \hat{e}^{245} + \hat{e}^{236} - \hat{e}^{146} + i(\hat{e}^{235} - \hat{e}^{145} - \hat{e}^{136} - \hat{e}^{246})\,)\,.$$

The exterior derivative of $e^{-2\hat{\varphi}}\hat{\Upsilon}$ is calculated as follows,

$$\begin{aligned}
d(e^{-2\hat{\varphi}}\hat{\Upsilon}) &= \frac{i}{\phi^4}(\,ih\partial_1 f_p + if_p(\phi\partial_1\phi - \psi\partial_2\phi) \\
&\quad -h\partial_2 f_p - f_p(\phi\partial_2\phi + \psi\partial_1\phi)\,)(\hat{e}^{1235} - i\hat{e}^{1245} - i\hat{e}^{1236} - \hat{e}^{1246})\,. \qquad \text{(D.36)}
\end{aligned}$$

From (D.11), we obtain

$$ih\partial_1 f_p + if_p(\phi\partial_1\phi - \psi\partial_2\phi) - h\partial_2 f_p - f_p(\phi\partial_2\phi + \psi\partial_1\phi) = 0 \qquad \text{(D.37)}$$

and this is rewritten as follows,

$$\partial_1 f_{pI} + f_{pI}\partial_1 \log(\phi^2 + \psi^2)^{\frac{1}{2}} + \partial_2 f_{pR} + f_{pR}\partial_2 \log(\phi^2 + \psi^2)^{\frac{1}{2}} = 0 \qquad \text{(D.38)}$$

$$\partial_1 f_{pR} + f_{pR}\partial_1 \log(\phi^2 + \psi^2)^{\frac{1}{2}} - \partial_2 f_{pI} + f_{pI}\partial_2 \log(\phi^2 + \psi^2)^{\frac{1}{2}} = 0\,, \qquad \text{(D.39)}$$

where f_{pR} denotes a real part of f_p and f_{pI} denotes an imaginary part of f_p. We see that the functions,

$$f_{pR} = \frac{\phi}{\sqrt{\phi^2 + \psi^2}}\ ,\ f_{pI} = \frac{\psi}{\sqrt{\phi^2 + \psi^2}}\,, \qquad \text{(D.40)}$$

are the solutions of the equations (D.38) and (D.39) and they satisfy $|f_p| = 1$. Therefore when $B_1 = B_2 = -\psi$, the deformed (3,0)-form $\hat{\Upsilon}$ takes the following form,

$$\hat{\Upsilon} = i\frac{\phi + i\psi}{\sqrt{\phi^2 + \psi^2}}(\,\hat{e}^{135} + \hat{e}^{245} + \hat{e}^{236} - \hat{e}^{146} + i(\hat{e}^{235} - \hat{e}^{145} - \hat{e}^{136} - \hat{e}^{246})\,)\,. \qquad \text{(D.41)}$$

D.1 Bismut curvature and Hull curvature

Curvature 2-forms of connection $\hat{\nabla}^{\pm}$ are defined by

$$\hat{\mathcal{R}}^{\pm}_{ab} = d\hat{\omega}^{\pm}_{ab} + \hat{\omega}^{\pm}_{ab} \wedge \hat{\omega}^{\pm}_{ab} = \frac{1}{2}\hat{R}^{\pm}_{ab\mu\nu}\hat{e}^{\mu\nu} , \tag{D.42}$$

where $\hat{R}^{\pm}_{ab\mu\nu}$ $(a, b, \mu, \nu = 1, \cdots, 6)$ denotes curvature components. In this section, we use the following notations

$$\Phi_1 = \partial_1 \log|\phi| , \ \Phi_2 = \partial_2 \log|\phi| , \ \Psi_1 = \partial_1^2 \log|\phi| , \ \Psi_2 = \partial_2^2 \log|\phi| , \ \Xi = \partial_2\partial_1 \log|\phi| . \tag{D.43}$$

The components of Bismut curvature are listed as follows:

$$\hat{R}^{+}_{1212} = -\frac{2}{\phi^4}(\Psi_1 + \Psi_2) , \ \hat{R}^{+}_{1234} = 0 , \ \hat{R}^{+}_{1256} = 0 ,$$

$$\hat{R}^{+}_{1313} = \frac{-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1) + \phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2 + \psi^2)^2} ,$$

$$\begin{aligned}\hat{R}^{+}_{1314} \;=\; & -\frac{\psi}{\phi^4(\phi^2 + \psi^2)^2}(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1) \\ & +\phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1)) ,\end{aligned}$$

$$\hat{R}^{+}_{1323} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2 + \psi^2) + \psi^3(-3\Phi_1^2 + \Phi_2^2 - \Psi_2) - \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2)}{\phi^3(\phi^2 + \psi^2)^2} ,$$

$$\hat{R}^{+}_{1324} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2 + \psi^2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4(\phi^2 + \psi^2)^2} ,$$

$$\hat{R}^{+}_{1413} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2 + \psi^2) + \psi^3(-\Phi_1^2 + 3\Phi_2^2 + \Psi_1) + \phi^2\psi(-3\Phi_1^2 + 5\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2 + \psi^2)^2} ,$$

$$\hat{R}^{+}_{1414} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2 + \psi^2) + \phi^2\psi(3\Phi_1^2 - 5\Phi_2^2 - \Psi_1) + \psi^3(\Phi_1^2 - 3\Phi_2^2 - \Psi_1))}{\phi^4\left(\phi^2 + \psi^2\right)^2} ,$$

$$\hat{R}^{+}_{1423} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2)}{\phi^3\left(\phi^2 + \psi^2\right)^2} ,$$

$$\hat{R}^{+}_{1424} = -\frac{\psi(\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4(\phi^2 + \psi^2)^2} ,$$

$$\hat{R}^{+}_{1515} = \frac{\Phi_1^2 - 3\Phi_2^2 - \Psi_1}{\phi^4} , \ \hat{R}^{+}_{1516} = 0 , \ \hat{R}^{+}_{1525} = -\frac{\Xi - 4\Phi_1\Phi_2}{\phi^4} , \ \hat{R}^{+}_{1526} = 0 ,$$

$$\hat{R}^{+}_{1615} = -\frac{\Xi - 4\Phi_1\Phi_2}{\phi^4} , \ \hat{R}^{+}_{1616} = 0 , \ \hat{R}^{+}_{1625} = \frac{-3\Phi_1^2 + \Phi_2^2 - \Psi_2}{\phi^4} , \ \hat{R}^{+}_{1626} = 0 ,$$

$$\hat{R}^{+}_{2313} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2 + \psi^2) + \psi^3(-\Phi_1^2 + 3\Phi_2^2 + \Psi_1) + \phi^2\psi(-3\Phi_1^2 + 5\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2 + \psi^2)^2} ,$$

$$\hat{R}^{+}_{2314} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(3\Phi_1^2 - 5\Phi_2^2 - \Psi_1) + \psi^3(\Phi_1^2 - 3\Phi_2^2 + \Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\hat{R}^{+}_{2323} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\begin{aligned}\hat{R}^{+}_{2324} &= -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2) \\ &\quad + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2)),\end{aligned}$$

$$\hat{R}^{+}_{2413} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 - \Psi_1) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 - \Psi_2)}{\phi^3\left(\phi^2+\psi^2\right)^2},$$

$$\hat{R}^{+}_{2414} = \frac{\psi(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1) + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\hat{R}^{+}_{2423} = \frac{6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\hat{R}^{+}_{2424} = -\frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4\left(\phi^2+\psi^2\right)^2},$$

$$\hat{R}^{+}_{2515} = -\frac{\Xi - 4\Phi_1\Phi_2}{\phi^4}\,,\ \hat{R}^{+}_{2516} = 0\,,\ \hat{R}^{+}_{2525} = \frac{-3\Phi_1^2 + \Phi_2^2 - \Psi_2}{\phi^4}\,,\ \hat{R}^{+}_{2526} = 0\,,$$

$$\hat{R}^{+}_{2615} = \frac{-\Phi_1^2 + 3\Phi_2^2 + \Psi_1}{\phi^4}\,,\ \hat{R}^{+}_{2616} = 0\,,\ \hat{R}^{+}_{2625} = \frac{\Xi - 4\Phi_1\Phi_2}{\phi^4}\,,\ \hat{R}^{+}_{2626} = 0\,,$$

$$\hat{R}^{+}_{3412} = \frac{\phi^2(\Phi_1^2 + \Phi_2^2) - \psi^2(\Psi_1 + \Psi_2)}{\phi^4(\phi^2+\psi^2)}\,,\ \hat{R}^{+}_{3434} = 0\,,\ \hat{R}^{+}_{3456} = 0\,,$$

$$\hat{R}^{+}_{3535} = \frac{\Phi_1^2 + \Phi_2^2}{\phi^4 + \phi^2\psi^2}\,,\ \hat{R}^{+}_{3536} = 0\,,\ \hat{R}^{+}_{3545} = -\frac{\psi(\Phi_1^2 + \Phi_2^2)}{\phi^3(\phi^2+\psi^2)}\,,\ \hat{R}^{+}_{3546} = 0\,,$$

$$\hat{R}^{+}_{3635} = -\frac{\psi(\Phi_1^2 + \Phi_2^2)}{\phi^3\left(\phi^2+\psi^2\right)}\,,\ 36\hat{3}6 = 0\,,\ 36\hat{4}5 = \frac{\psi^2(\Phi_1^2 + \Phi_2^2)}{\phi^4(\phi^2+\psi^2)}\,,\ \hat{R}^{+}_{3646} = 0\,,$$

$$\hat{R}^{+}_{4535} = \frac{\psi(\Phi_1^2 + \Phi_2^2)}{\phi^3\left(\phi^2+\psi^2\right)}\,,\ \hat{R}^{+}_{4536} = 0\,,\ \hat{R}^{+}_{4545} = -\frac{\psi^2(\Phi_1^2 + \Phi_2^2)}{\phi^4(\phi^2+\psi^2)}\,,\ \hat{R}^{+}_{4546} = 0\,,$$

$$\hat{R}^{+}_{4635} = \frac{(\Phi_1^2 + \Phi_2^2)}{\phi^2\left(\phi^2+\psi^2\right)}\,,\ \hat{R}^{+}_{4636} = 0\,,\ \hat{R}^{+}_{4645} = -\frac{\psi(\Phi_1^2 + \Phi_2^2)}{\phi^3(\phi^2+\psi^2)}\,,\ \hat{R}^{+}_{4646} = 0\,,$$

$$\hat{R}^{+}_{5612} = -\frac{\Psi_1 + \Psi_2}{\phi^4}\,,\ \hat{R}^{+}_{5634} = 0\,,\ 56\hat{5}6 = 0\,.$$

One can notice the relations

$$-\hat{R}^{+}_{12} + \hat{R}^{+}_{34} + \hat{R}^{+}_{56} = 0\,,\ \hat{R}^{+}_{13} = -\hat{R}^{+}_{24}\,,\ \hat{R}^{+}_{14} = \hat{R}^{+}_{23}\,,\ \hat{R}^{+}_{15} = -\hat{R}^{+}_{26}\,,\ \hat{R}^{+}_{16} = \hat{R}^{+}_{25}\,. \tag{D.44}$$

Also, the components of Hull curvature are listed as follows:

$$\hat{R}^{-}_{1212} = -\frac{2}{\phi^4}(\Psi_1 + \Psi_2)\,,\ \hat{R}^{-}_{1234} = \frac{(\Phi_1 + \Phi_2)}{\phi^4}\,,\ \hat{R}^{+}_{1256} = \frac{(\Phi_1 + \Phi_2)}{\phi^4}\,,$$

$$\hat{R}^{-}_{1313} = \frac{-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1) + \phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{1314} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) + \psi^3(-\Phi_1^2 + 3\Phi_2^2 + \Psi_1) + \phi^2\psi(-3\Phi_1^2 + 5\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{1323} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) - \psi^3(2\Phi_1^2 - 2\Phi_2^2 + \Psi_2) - \phi^2\psi(4\Phi_1^2 - 4\Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{1324} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(4\Phi_1^2 - 2\Phi_2^2 + \Psi_2) + \phi\psi^2(2\Phi_1^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\begin{aligned}\hat{R}^{-}_{1413} \;=\;& -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1)\\ &+\phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1))\,,\end{aligned}$$

$$\hat{R}^{-}_{1414} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(3\Phi_1^2 - 5\Phi_2^2 - \Psi_1) + \psi^3(\Phi_1^2 - 3\Phi_2^2 - \Psi_1))}{\phi^4(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{1423} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(4\Phi_1^2 - 4\Phi_2^2 - \Psi_2) + \psi^3(2\Phi_1^2 - 2\Phi_2^2 + \Psi_2))}{\phi^4(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{1424} = -\frac{\psi(\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(4\Phi_1^2 - 2\Phi_2^2 + \Psi_2) + \phi\psi^2(2\Phi_1^2 + \Psi_2))}{\phi^4(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{1515} = \frac{\Phi_1^2 - 3\Phi_2^2 - \Psi_1}{\phi^4}\,,\ \hat{R}^{-}_{1516} = -\frac{\Xi - 4\Phi_1\Phi_2}{\phi^4}\,,$$

$$\hat{R}^{-}_{1525} = -\frac{\Xi - 4\Phi_1\Phi_2}{\phi^4}\,,\ \hat{R}^{-}_{1526} = -\frac{2\Phi_1^2 - 2\Phi_2^2 + \Psi_2}{\phi^4}\,,$$

$$\hat{R}^{-}_{1615} = \hat{R}^{-}_{1616} = \hat{R}^{-}_{1625} = \hat{R}^{-}_{1626} = 0\,,$$

$$\hat{R}^{-}_{2313} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) + \psi^3(-2\Phi_1^2 + 2\Phi_2^2 + \Psi_1) + \phi^2\psi(-4\Phi_1^2 + 4\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{2314} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(2\Phi_1^2 - 4\Phi_2^2 - \Psi_1) - \phi\psi^2(2\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{2323} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{2324} = \frac{6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\hat{R}^{-}_{2413} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(4\Phi_1^2 - 4\Phi_2^2 - \Psi_1) + \psi^3(2\Phi_1^2 - 2\Phi_2^2 - \Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\hat{R}^{-}_{2414} = \frac{\psi(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(-2\Phi_1^2 + 4\Phi_2^2 + \Psi_1) + \phi\psi^2(2\Phi_2^2 + \Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\begin{aligned}\hat{R}^{-}_{2423} &= -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2) \\ &\quad + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2)),\end{aligned}$$

$$\hat{R}^{-}_{2424} = -\frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4\left(\phi^2+\psi^2\right)^2},$$

$$\hat{R}^{-}_{2515} = -\frac{\Xi - 4\Phi_1\Phi_2}{\phi^4}\ ,\ \hat{R}^{-}_{2516} = \frac{-2\Phi_1^2 + 2\Phi_2^2 + \Psi_1}{\phi^4}\ ,$$
$$\hat{R}^{-}_{2525} = \frac{-3\Phi_1^2 + \Phi_2^2 - \Psi_2}{\phi^4}\ ,\ \hat{R}^{+}_{2526} = \frac{\Xi - 4\Phi_1\Phi_2}{\phi^4},$$

$$\hat{R}^{-}_{2615} = \hat{R}^{-}_{2616} = \hat{R}^{-}_{2625} = \hat{R}^{-}_{2626} = 0\ \ ,\ \ \hat{R}^{-}_{3412} = \hat{R}^{-}_{3434} = \hat{R}^{-}_{3456} = 0,$$

$$\hat{R}^{-}_{3535} = \frac{\Phi_1^2 + \Phi_2^2}{\phi^4 + \phi^2\psi^2}\ ,\ \hat{R}^{-}_{3536} = -\frac{\psi(\Phi_1^2 + \Phi_2^2)}{\phi^3(\phi^2+\psi^2)},$$
$$\hat{R}^{-}_{3545} = \frac{\psi(\Phi_1^2 + \Phi_2^2)}{\phi^3(\phi^2+\psi^2)}\ ,\ \hat{R}^{-}_{3546} = \frac{(\Phi_1^2 + \Phi_2^2)}{\phi^2(\phi^2+\psi^2)},$$

$$\hat{R}^{-}_{3635} = \hat{R}^{-}_{3636} = \hat{R}^{-}_{3645} = \hat{R}^{-}_{3646} = 0,$$

$$\hat{R}^{-}_{4535} = -\frac{\Phi_1^2 + \Phi_2^2}{\phi^4 + \phi^2\psi^2}\ ,\ \hat{R}^{-}_{4536} = \frac{\psi^2(\Phi_1^2 + \Phi_2^2)}{\phi^4(\phi^2+\psi^2)},$$
$$\hat{R}^{-}_{4545} = -\frac{\psi^2(\Phi_1^2 + \Phi_2^2)}{\phi^4(\phi^2+\psi^2)}\ ,\ \hat{R}^{-}_{4546} = -\frac{(\Phi_1^2 + \Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)},$$

$$\hat{R}^{-}_{4635} = \hat{R}^{-}_{4636} = \hat{R}^{-}_{4645} = \hat{R}^{-}_{4646} = 0\ ,\ \hat{R}^{-}_{5612} = \hat{R}^{-}_{5634} = \hat{R}^{-}_{5656} = 0.$$

It should be notice that well-known identity $\hat{R}^{+}_{ab\mu\nu} - \hat{R}^{-}_{\mu\nu ab} = d\hat{T}_{ab\mu\nu} = 0$ is valid.

Appendix E

SU(3) structure of the T-dualized solution along ∂_5

We T-dualized the solutions (4.50), (4.51), and (4.52) along ∂_5 and then take the following form:

$$\begin{aligned}
\tilde{g}_6 &= \frac{1}{\phi^2+\psi^2}(d\tilde{x}^5 - B_2 dx^6) + \phi^4((dx^1)^2+(dx^2)^2) + \frac{1}{\phi^2+\psi^2}(d\hat{x}^3+\psi dx^4)^2 \\
&\quad + \frac{\phi^2}{\phi^2+\psi^2}(\,(dx^4)^2+(dx^6)^2\,)\,, && \text{(E.1)} \\
& && \text{(E.2)} \\
\tilde{B} &= -\frac{\psi}{\phi^2+\psi^2}dx^6\wedge d\tilde{x}^5 + \frac{\psi}{\phi^2+\psi^2}d\hat{x}^3\wedge dx^4\,, && \text{(E.3)} \\
e^{2\tilde{\varphi}} &= \frac{1}{\phi^2+\psi^2}e^{2\hat{\varphi}} = \frac{1}{(\phi^2+\psi^2)^2}\phi^2\,. && \text{(E.4)}
\end{aligned}$$

The orthonormal basis is defined by Let us introduce an new orthonormal basis;

$$\begin{aligned}
&\tilde{e}^1 = \phi^2 dx^1\,,\ \tilde{e}^2 = \phi^2 dx^2\,,\ \tilde{e}^3 = \frac{1}{\sqrt{\phi^2+\psi^2}}(d\hat{x}^3+\psi dx^4)\,,\ \tilde{e}^4 = \frac{\phi}{\sqrt{\phi^2+\psi^2}}dx^4 \\
&\tilde{e}^5 = \frac{1}{\sqrt{\phi^2+\psi^2}}(d\tilde{x}^5 - B_2 dx^6)\,,\ \tilde{e}^6 = \frac{\phi}{\sqrt{\phi^2+\psi^2}}dx^6 && \text{(E.5)}
\end{aligned}$$

and exterior derivatives of them are the following:

$$\begin{aligned}
d\tilde{e}^1 &= -\frac{2}{\phi^3}\frac{\partial\phi}{\partial x^2}\tilde{e}^{12}\,,\ d\tilde{e}^2 = \frac{2}{\phi^3}\frac{\partial\phi}{\partial x^1}\tilde{e}^{12}\,, \\
d\tilde{e}^3 &= -\frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^{13} + \frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_2\phi+\psi\partial_1\phi)\tilde{e}^{23} - \frac{1}{\phi^3}\partial_2\phi\tilde{e}^{14} + \frac{1}{\phi^3}\partial_1\phi\tilde{e}^{24}\,, \\
d\tilde{e}^4 &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\psi\partial_1\phi+\phi\partial_2\phi)\tilde{e}^{14} + (\psi\partial_2\phi - \phi\partial_1\phi)\tilde{e}^{24}\,)\,, \\
d\tilde{e}^5 &= \frac{1}{\phi^2(\phi^2+\psi^2)}(\,-(\phi\partial_1\phi-\psi\partial_2\phi)\tilde{e}^{15} - (\phi\partial_2\phi+\psi\partial_1\phi)\tilde{e}^{25}\,) - \frac{1}{\phi^3}\partial_1 B_2\tilde{e}^{16} - \frac{1}{\phi^3}\partial_2 B_2\tilde{e}^{26}\,, \\
d\tilde{e}^6 &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\,(\psi\partial_1\phi+\phi\partial_2\phi)\tilde{e}^{16} + (\psi\partial_2\phi-\phi\partial_1\phi)\tilde{e}^{26}\,)\,. && \text{(E.6)}
\end{aligned}$$

Thus we obtain the function $\tilde{f}^\mu{}_{\nu\rho}$ that are defined by $d\tilde{e}^\mu = \frac{1}{2}\tilde{f}^\mu{}_{\nu\rho}\tilde{e}^{\nu\rho}$ as follows;

$$
\begin{aligned}
\tilde{f}^1{}_{12} &= -\frac{2}{\phi^3}\partial_2\phi\,,\ \tilde{f}^2{}_{12} = \frac{2}{\phi^3}\partial_1\phi\,,\\
\tilde{f}^3{}_{13} &= -\frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_1\phi - \psi\partial_2\phi)\,,\ \tilde{f}^3{}_{23} = -\frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_2\phi + \psi\partial_1\phi)\,,\\
\tilde{f}^3{}_{14} &= -\frac{1}{\phi^3}\partial_2\phi\,,\ \tilde{f}^3{}_{24} = \frac{1}{\phi^3}\partial_1\phi\,,\\
\tilde{f}^4{}_{14} &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\psi\partial_1\phi + \phi\partial_2\phi)\,,\ \tilde{f}^4{}_{24} = \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\psi\partial_2\phi - \phi\partial_1\phi)\,,\\
\tilde{f}^5{}_{15} &= -\frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_1\phi - \psi\partial_2\phi)\,,\ f^{\tilde{5}}{}_{25} = -\frac{1}{\phi^2(\phi^2+\psi^2)}(\phi\partial_2\phi + \psi\partial_1\phi)\,,\\
\tilde{f}^5{}_{16} &= -\frac{1}{\phi^3}\partial_1 B_2\,,\ \tilde{f}^5{}_{26} = -\frac{1}{\phi^3}\partial_2 B_2\,,\\
\tilde{f}^6{}_{16} &= \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\psi\partial_1\phi + \phi\partial_2\phi)\,,\ \tilde{f}^6{}_{26} = \frac{\psi}{\phi^3(\phi^2+\psi^2)}(\psi\partial_2\phi - \phi\partial_1\phi)\,.
\end{aligned}
\tag{E.7}
$$

A flux of the B-field $\tilde{B}$ is derived by

$$\tilde{H} = -d\tilde{B} \tag{E.8}$$

and thus the flux $\tilde{H}$ is explicitly given by

$$
\begin{aligned}
\tilde{H} &= \frac{1}{\phi^3(\phi^2+\psi^2)}\Big(\big((\phi^2-\psi^2)\partial_2\phi + 2\phi\psi\partial_1\phi\big)\,\tilde{e}^{134} - \big((\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi\big)\,\tilde{e}^{234}\\
&\quad - \big((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi\big)\,\tilde{e}^{156} - \big((\phi^2-\psi^2)\partial_1\phi - 2\phi\psi\partial_2\phi\big)\,\tilde{e}^{256}\Big)\,.
\end{aligned}
\tag{E.9}
$$

We should investigate that the triplet $(\tilde{g}, \tilde{H}, \tilde{\varphi})$ satisfies supersymmetry preserving conditions:

$$d(e^{-2\tilde{\varphi}} * \tilde{\kappa}) = 0\,, \tag{E.10}$$

$$d(e^{-2\tilde{\varphi}}\tilde{\Upsilon}) = 0\,, \tag{E.11}$$

$$*\tilde{H} = e^{2\tilde{\varphi}} d(e^{-2\tilde{\varphi}}\tilde{\kappa})\,, \tag{E.12}$$

where $\tilde{J}$, $\tilde{\kappa}$, $\tilde{\Upsilon}^{(3,0)}$ represent an almost complex structure, a fundamental 2-form, and a (3,0)-form respectively.
We assume a fundamental 2-form as follows,

$$\tilde{\kappa} = \tilde{K}(x^1, x^2)(\alpha_1\tilde{e}^{12} + \alpha_2\tilde{e}^{34} + \alpha_3\tilde{e}^{56})\,. \tag{E.13}$$

Let us determine a function $\tilde{K}(x^1, x^2)$ such that $\tilde{\kappa}$ satisfies (E.10). We define a volume form to the metric (E.2) by

$$\text{Vol.} = *1\,,\ \text{Vol.} = \tilde{e}^{123456}\,. \tag{E.14}$$

Then Hodge dual of $\tilde{e}^{12}$, $\tilde{e}^{34}$ and $\tilde{e}^{56}$ are given by

$$*\tilde{e}^{12} = \iota_{\tilde{e}_2}\iota_{\tilde{e}_1}\text{Vol.} = \tilde{e}^{3456}\,,\ *\tilde{e}^{34} = \tilde{e}^{1256}\,,\ *\tilde{e}^{56} = \tilde{e}^{1234}\,.$$

We have

$$*\tilde{\kappa} = \alpha_1 \tilde{e}^{3456} + \alpha_2 \tilde{e}^{1256} + \alpha_3 \tilde{e}^{1234} \tag{E.15}$$

and

$$e^{-2\tilde{\varphi}} * \tilde{\kappa} = \frac{h^2}{\phi^2}(\alpha_1 \tilde{e}^{3456} + \alpha_2 \tilde{e}^{1256} + \alpha_3 \tilde{e}^{1234}),$$

where

$$h = \phi^2 + \psi^2 \tag{E.16}$$

and

$$dh = \frac{2}{\phi^2}((\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^1 + (\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^2). \tag{E.17}$$

The left hand side of (E.10) takes the form

$$\begin{aligned} d(e^{-2\tilde{\varphi}} * \tilde{\kappa}) &= \frac{2h\tilde{K}}{\phi^2} dh \wedge (\alpha_1 \tilde{e}^{3456} + \alpha_2 \tilde{e}^{1256} + \alpha_3 \tilde{e}^{1234}) - \frac{h^2\tilde{K}}{\phi^4} d(\phi^2) \wedge (\alpha_1 \tilde{e}^{3456} + \alpha_2 \tilde{e}^{1256} + \alpha_3 \tilde{e}^{1234}) \\ &+ \frac{h^2}{\phi^2} d\tilde{K} \wedge (\alpha_1 \tilde{e}^{3456} + \alpha_2 \tilde{e}^{1256} + \alpha_3 \tilde{e}^{1234}) + \frac{h^2\tilde{K}}{\phi^2}(\alpha_1 d\tilde{e}^{3456} + \alpha_2 d\tilde{e}^{1256} + \alpha_3 d\tilde{e}^{1234}) \end{aligned} \tag{E.18}$$

and each terms of (E.18) are the following:

$$1st\, term = \alpha_1 \frac{4h\tilde{K}\phi^2}{\phi^6}((\phi\partial_1\phi - \psi\partial_2\phi)\tilde{e}^{13456} + (\phi\partial_2\phi + \psi\partial_1\phi)\tilde{e}^{23456})$$

$$2nd\, term = -\alpha_1 \frac{h^2\tilde{K}}{\phi^6}(\partial_1(\phi^2)\tilde{e}^{13456} + \partial_2(\phi^2)\tilde{e}^{23456})$$

$$3rd\, term = \alpha_1 \frac{h^2}{\phi^6}(\phi^2(\partial_1\tilde{K})\tilde{e}^{13456} + \phi^2(\partial_2\tilde{K})\tilde{e}^{23456})$$

$$4th\, term = \alpha_1 \frac{h^2\tilde{K}}{\phi^6}(\phi^4(\tilde{f}^3{}_{13} + \tilde{f}^4{}_{14} + \tilde{f}^5{}_{15} + \tilde{f}^6{}_{16})\tilde{e}^{13456} + \phi^4(\tilde{f}^3{}_{23} + \tilde{f}^4{}_{24} + \tilde{f}^5{}_{25} + \tilde{f}^6{}_{26})\tilde{e}^{23456}).$$

Then, we have

$$\begin{aligned} d(e^{-2\tilde{\varphi}} * \tilde{\kappa}) &= \frac{\alpha_1}{\phi^6}(\, h^2(\phi^2(\partial_1\tilde{K}) + \phi^4\tilde{K}(\tilde{f}^3{}_{13} + \tilde{f}^4{}_{14} + \tilde{f}^5{}_{15} + \tilde{f}^6{}_{16})) - h^2\tilde{K}\partial_1(\phi^2) \\ &+ 2\tilde{K}\phi^2(h(\phi\partial_1\phi - \psi\partial_2\phi) + h(\phi\partial_1\phi - \psi\partial_2\phi))]\tilde{e}^{13456} \\ &+ \frac{\alpha_1}{\phi^6}[h^2(\phi^2(\partial_2\tilde{K}) + \phi^4\tilde{K}(\tilde{f}^3{}_{23} + \tilde{f}^4{}_{24} + \tilde{f}^5{}_{25} + \tilde{f}^6{}_{26})) - h^2\tilde{K}\partial_2(\phi^2) \\ &+ 2\tilde{K}\phi^2(h(\phi\partial_2\phi + \psi\partial_1\phi) + h(\phi\partial_2\phi + \psi\partial_1\phi))\,)\tilde{e}^{23456}, \end{aligned}$$

where

$$\begin{aligned} h^2\phi^4(\tilde{f}^3{}_{13} + \tilde{f}^4{}_{14} + \tilde{f}^5{}_{15} + \tilde{f}^6{}_{16}) &= h\phi(\,(\psi^2 - \phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi\,) - h\phi^2(\phi\partial_1\phi - \psi\partial_2\phi) \\ &+ h\phi\psi^2\partial_1\phi + h\phi^2\psi\partial_2\phi\,, \end{aligned}$$

$$
\begin{aligned}
h^2\phi^4(\tilde{f}^3{}_{23}+\tilde{f}^4{}_{24}+\tilde{f}^5{}_{25}+\tilde{f}^6{}_{26}) &= h\phi(\,(\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi\,)-h\phi^2(\phi\partial_2\phi+\psi\partial_1\phi)\\
&\quad +h\phi\psi^2\partial_2\phi-h\phi^2\psi\partial_1\phi\,.
\end{aligned}
$$

Thus left hand side of (E.18) is

$$
d(e^{-2\tilde{\varphi}}*\tilde{\kappa}) \;=\; \frac{\alpha_1}{\phi^6}h\phi^2(\,(\partial_1\tilde{K})\tilde{e}^{13456}+(\partial_2\tilde{K})\tilde{e}^{23456}\,)
$$

and thus we obtain the following equations for the function $\tilde{K}(x^1,x^2)$ by (E.10)

$$
\partial_1\tilde{K}=0\;,\;\partial_2\tilde{K}=0\,.
$$

Then, the function $\tilde{K}(x^1,x^2)=1$. Consequently, the fundamental 2-form $\tilde{\kappa}$ takes the form

$$
\tilde{\kappa}=\alpha_1\tilde{e}^{12}+\alpha_2\tilde{e}^{34}+\alpha_3\tilde{e}^{56}\,. \tag{E.19}
$$

From (E.19), an almost complex structure $\tilde{J}$ is defined by

$$
\tilde{J}\tilde{e}_1=-\alpha_1\tilde{e}_2\;,\;\tilde{J}\tilde{e}_3=-\alpha_2\tilde{e}\;,\;\tilde{J}\tilde{e}_5=-\alpha_3\tilde{e}_6 \tag{E.20}
$$

and $\tilde{J}_*\tilde{e}^\mu\,(\mu=1,\cdots 6)$ are given by

$$
\tilde{J}_*\tilde{e}_1=\alpha_1\tilde{e}_2\;,\;\tilde{J}_*\tilde{e}_3=\alpha_2\tilde{e}_4\;,\;\tilde{J}_*\tilde{e}_5=\alpha_3\tilde{e}_6\,. \tag{E.21}
$$

A Nijenhuis tensor N associated with the almost complex structure $\tilde{J}$ is defined by

$$
N_{\tilde{J}}(X,Y)=[\tilde{J}X\,,\,\tilde{J}Y]-[X\,,\,Y]-\tilde{J}[\tilde{J}X\,,\,Y]-\tilde{J}[X\,,\,\tilde{J}Y]\,, \tag{E.22}
$$

where a commutation relation between vector fields are given by $[\tilde{e}_\mu\,,\,\tilde{e}_\nu]=-\tilde{f}^\rho{}_{\mu\nu}\tilde{e}_\rho$. We require $\alpha_1\alpha_2=-1$ and $\alpha_1\alpha_3=-1$ and then we have

$$
\begin{aligned}
N_{\tilde{J}}(\tilde{e}_1,\tilde{e}_5) &= \frac{1}{\phi^3}(\,-(\partial_1\phi+\partial_2B_2)\tilde{e}_5+(\partial_2\phi-\partial_1B_2)\tilde{e}_6\,)\,,\\
N_{\tilde{J}}(\tilde{e}_2,\tilde{e}_5) &= \frac{1}{\phi^3}(\,(-\partial_2\phi+\partial_1B_2)\tilde{e}_5-(\partial_1\phi+\partial_2B_2)\tilde{e}_6\,)\,,\\
N_{\tilde{J}}(\tilde{e}_1,\tilde{e}_6) &= \frac{1}{\phi^3}(\,(\partial_2\phi-\partial_1B_2)\tilde{e}_5+(\partial_1\phi+\partial_2B_2)\tilde{e}_6\,)\,,\\
N_{\tilde{J}}(\tilde{e}_2,\tilde{e}_6) &= -\frac{1}{\phi^3}(\,(\partial_1\phi+\partial_2B_2)\tilde{e}_5-(-\partial_2\phi+\partial_1B_2)\tilde{e}_6\,)\,,
\end{aligned} \tag{E.23}
$$

If $B_2=-\psi$, then from Cauchy–Riemann equations (4.20) the components of the Nijenhuis tensor are the following,

$$
N_{\tilde{J}}(\tilde{e}_1,\tilde{e}_5)=0\;,\;N_{\tilde{J}}(\tilde{e}_2,\tilde{e}_5)=0\;,\;N_{\tilde{J}}(\tilde{e}_1,\tilde{e}_6)=0\;,\;N_{\tilde{J}}(\tilde{e}_2,\tilde{e}_6)=0\,. \tag{E.24}
$$

The rest of $N_{\tilde{J}}(\tilde{e}_\mu,\tilde{e}_\nu)$ are clearly zero by using Cauchy–Riemann conditions. Thus an almost complex structure $\tilde{J}$ is a complex structure when $B_2=-\tilde{\psi}$.

A Bismut torsion associated with $\tilde{J}$ is defined by

$$\tilde{T} := \alpha_4 \tilde{J}_* d\tilde{\kappa} \,. \tag{E.25}$$

The exterior derivative of $\tilde{\kappa}$ is given by

$$\begin{aligned} d\tilde{\kappa} &= \frac{\alpha_2}{\phi^3(\phi^2+\psi^2)}\big(((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)\tilde{e}^{134} + ((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)\tilde{e}^{234} \big) \\ &\quad + \frac{\alpha_3}{\phi^3(\phi^2+\psi^2)}\big(((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)\tilde{e}^{156} + ((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)\tilde{e}^{256} \big) \end{aligned} \tag{E.26}$$

and thus we have

$$\begin{aligned} \tilde{J}_* d\tilde{\kappa} &= -\frac{1}{\phi^3(\phi^2+\psi^2)}\big(((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)\tilde{e}^{234} - ((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)\tilde{e}^{134} \\ &\quad + ((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)\tilde{e}^{256} - ((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)\tilde{e}^{156} \big) \,. \end{aligned}$$

Therefore the Bismut torsion $\tilde{T}$ takes the following form,

$$\begin{aligned} \tilde{T} &= -\frac{\alpha_4}{\phi^3(\phi^2+\psi^2)}\Big(((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)\tilde{e}^{234} - ((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)\tilde{e}^{134} \\ &\quad + ((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)\tilde{e}^{256} - ((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)\tilde{e}^{156} \Big) \end{aligned} \tag{E.27}$$

This is obviously coincident with (E.9) under the condition $\alpha_4 = -1$,

$$\tilde{H} = \tilde{T} \quad \text{for } \alpha_4 = -1 \,. \tag{E.28}$$

Let us check whether the torsion $\tilde{T}$ with $\alpha_4 = -1$ satisfies the equation $d^{\tilde{T}}\tilde{\kappa} = 0$ or not. According to Ref. [57], an exterior derivative with a torsion for the fundamental 2-form $\tilde{\kappa}$,

$$d^{\tilde{T}}\tilde{\kappa} = d\tilde{\kappa} - \sum_{\mu=1}^{6} (\iota_{\tilde{e}_\mu}\tilde{T}) \wedge (\iota_{\tilde{e}_\mu}\tilde{\kappa}) \,. \tag{E.29}$$

The first term of $d^{\tilde{T}}\hat{\kappa}$ has been given by (E.26). On the second term, $(\iota_{\tilde{e}_\mu}\tilde{T}) \wedge (\iota_{\tilde{e}_\mu}\tilde{\kappa})$ for each values of $\mu = 1, \cdots, 6$ are given by

$$\begin{aligned} (\iota_{\tilde{e}_1}\tilde{T}) \wedge (\iota_{\tilde{e}_1}\tilde{\kappa}) &= \frac{\alpha_1\alpha_4}{\phi^3(\phi^2+\psi^2)}((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)(\tilde{e}^{234} + \tilde{e}^{256}) \,, \\ (\iota_{\tilde{e}_2}\tilde{T}) \wedge (\iota_{\tilde{e}_2}\tilde{\kappa}) &= \frac{\alpha_1\alpha_4}{\phi^3(\phi^2+\psi^2)}((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)(\tilde{e}^{134} + \tilde{e}^{156}) \,, \\ (\iota_{\tilde{e}_3}\tilde{T}) \wedge (\iota_{\tilde{e}_3}\tilde{\kappa}) &= 0 \,,\ (\iota_{\tilde{e}_4}\tilde{T}) \wedge (\iota_{\tilde{e}_4}\tilde{\kappa}) = 0 \,,\ (\iota_{\tilde{e}_5}\tilde{T}) \wedge (\iota_{\tilde{e}_5}\tilde{\kappa}) = 0 \,,\ (\iota_{\tilde{e}_6}\tilde{T}) \wedge (\iota_{\tilde{e}_6}\tilde{\kappa}) = 0 \end{aligned}$$

and thus we have

$$\begin{aligned} \sum_{\mu=1}^{6} (\iota_{\tilde{e}_\mu}\tilde{T}) \wedge (\iota_{\tilde{e}_\mu}\tilde{\kappa}) &= \frac{\alpha_1\alpha_4}{\phi^3(\phi^2+\psi^2)}\big(((\psi^2-\phi^2)\partial_1\phi + 2\phi\psi\partial_2\phi)(\tilde{e}^{134} + \tilde{e}^{156}) \\ &\quad + ((\psi^2-\phi^2)\partial_2\phi - 2\phi\psi\partial_1\phi)(\tilde{e}^{234} + \tilde{e}^{256}) \big) \,. \end{aligned}$$

Therefore $d^{\tilde{T}}\tilde{\kappa}$ is given by

$$\begin{aligned}
d^{\tilde{T}}\tilde{\kappa} &= \frac{\alpha_2}{\phi^3(\phi^2+\psi^2)}\big(\,((\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi)\tilde{e}^{134}+((\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi)\tilde{e}^{234}\,\big) \\
&\quad+\frac{\alpha_3}{\phi^3(\phi^2+\psi^2)}\big(\,((\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi)\tilde{e}^{156}+((\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi)\tilde{e}^{256}\,\big) \\
&\quad-\frac{\alpha_1\alpha_4}{\phi^3(\phi^2+\psi^2)}\big(\,((\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi)(\tilde{e}^{134}+\tilde{e}^{156}) \\
&\quad+((\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi\}(\tilde{e}^{234}+\tilde{e}^{256})\,\big)
\end{aligned}$$

and thus we see that $d^{\tilde{T}}\tilde{\kappa}=0$ when $\alpha_1\alpha_2=-1$, $\alpha_1\alpha_3=-1$ and $\alpha_4=-1$.

A Lee form $\tilde{\Theta}$ is defined by

$$\tilde{\Theta}=\alpha_4\tilde{J}_*\delta\tilde{\kappa}\,, \tag{E.30}$$

where the operator δ is a co-derivative on p-form that is defined by $\delta=(-1)^{np-p^2+p}*d*$. The exterior derivatives of $*\tilde{\kappa}$ is given by

$$d*\tilde{\kappa}=\frac{2\alpha_1}{\phi^3(\phi^2+\psi^2)}\big(\,(\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi\,\big)\tilde{e}^{13456}+\frac{2\alpha_1}{\phi^3(\phi^2+\psi^2)}\big(\,-2\phi\psi\partial_1\phi+(\psi^2-\phi^2)\partial_2\phi\,\big)\tilde{e}^{23456} \tag{E.31}$$

and thus the Lee form takes the following form

$$\tilde{\Theta}=-\alpha_4 d\left(\log\frac{\phi^2}{(\phi^2+\psi^2)^2}\right). \tag{E.32}$$

If we identify a dilaton with the Lee form by $d\tilde{\varphi}=-\frac{1}{2}\alpha_4\tilde{\Theta}$, the dilaton $\tilde{\varphi}$ is given by

$$\tilde{\varphi}=\frac{1}{2}\log\frac{\phi^2}{(\phi^2+\psi^2)^2} \tag{E.33}$$

and this is consistent with (4.71).

We consider (E.12).The Hodge dual of $*\tilde{H}$ is given by

$$\begin{aligned}
*\tilde{H} &= \frac{1}{\phi^3(\phi^2+\psi^2)}\Big(\,\big((\phi^2-\psi^2)\partial_2\phi+2\phi\psi\partial_1\phi\big)\,\tilde{e}^{256}+\big((\phi^2-\psi^2)\partial_1\phi-2\phi\psi\partial_2\phi\big)\,\tilde{e}^{156}\,\Big) \\
&\quad+\frac{1}{\phi^3(\phi^2+\psi^2)}\Big(-\big((\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi\big)\,\tilde{e}^{234}+\big((\phi^2-\psi^2)\partial_1\phi-2\phi\psi\partial_2\phi\big)\,\tilde{e}^{134}\,\Big)
\end{aligned} \tag{E.34}$$

The right hand side of (E.12) becomes

$$\begin{aligned}
e^{2\tilde{\varphi}}d(e^{-2\tilde{\varphi}}\tilde{\kappa}) &= d\tilde{\kappa}-\tilde{\Theta}\wedge\tilde{\kappa} \\
&= -\frac{\alpha_2}{\phi^3(\phi^2+\psi^2)}\big(\,((\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi)\tilde{e}^{134}+((\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi)\tilde{e}^{234}\,\big) \\
&\quad-\frac{\alpha_3}{\phi^3(\phi^2+\psi^2)}\big(\,((\psi^2-\phi^2)\partial_1\phi+2\phi\psi\partial_2\phi)\tilde{e}^{156}+((\psi^2-\phi^2)\partial_2\phi-2\phi\psi\partial_1\phi)\tilde{e}^{256}\,\big)\,.
\end{aligned}$$

If $\alpha_2 = \alpha_3 = 1$, we obtain

$$*H - e^{2\tilde{\varphi}} d(e^{-2\tilde{\varphi}}\tilde{\kappa}) = 0\,. \tag{E.35}$$

Namely, the triplet $(\tilde{g}, \tilde{H}, \tilde{\varphi})$ satisfies (E.12) under the condition $\alpha_2 = \alpha_3 = 1$.

We consider a $(3,0)$-form $\tilde{\Upsilon}$ which satisfies (E.11).We assume that $\tilde{\Upsilon}$ takes the form

$$\begin{aligned}\tilde{\Upsilon} &= \sqrt{\alpha_1\alpha_2\alpha_3}\tilde{W}(x^1,x^2)(\tilde{e}^1 + i\tilde{e}^2)\wedge(\tilde{e}^3 - i\tilde{e}^4)\wedge(\tilde{e}^5 - i\tilde{e}^6) \\ &= \sqrt{\alpha_1\alpha_2\alpha_3}\tilde{W}(x^1,x^2)(\,\hat{e}^{135} + \tilde{e}^{245} + \tilde{e}^{236} - \tilde{e}^{146} + i(\tilde{e}^{235} - \tilde{e}^{145} - \tilde{e}^{136} - \tilde{e}^{246})\,)\,,\end{aligned} \tag{E.36}$$

where the function $\tilde{W}$ is a complex function. A $(3,0)$-form $\tilde{\Upsilon}$ has the relation between the fundamental 2-form $\tilde{\kappa}$ by the volume matching condition,

$$\tilde{\Upsilon}\wedge\bar{\tilde{\Upsilon}} = \frac{4}{3} i\,\tilde{\kappa}^3\,. \tag{E.37}$$

Calculating (E.37) directly, we obtain the conditions

$$\alpha_1\alpha_2\alpha_3 = -1\;,\;|\tilde{W}|^2 = 1 \tag{E.38}$$

and thus the complex function $\tilde{W}$ is a phase function,

$$\tilde{W}(x^1,x^2) = f_p(x^1,x^2)\,, \tag{E.39}$$

where $|f_p| = 1$. We determine f_p so as to satisfy (E.11). $e^{-2\tilde{\varphi}}\tilde{\Upsilon}$ is given by

$$e^{-2\tilde{\varphi}}\tilde{\Upsilon} = i\,\frac{h^2 f_p}{\phi^2}(\,\tilde{e}^{135} + \tilde{e}^{245} + \tilde{e}^{236} - \tilde{e}^{146} + i(\tilde{e}^{235} - \tilde{e}^{145} - \tilde{e}^{136} - \tilde{e}^{246})\,)\,.$$

The exterior derivative of $e^{-2\tilde{\varphi}}\tilde{\Upsilon}$ is calculated as follows,

$$\begin{aligned}d(e^{-2\tilde{\varphi}}\tilde{\Upsilon}) &= \frac{ih}{\phi^5}(\,-(h\phi\partial_2 f_p + 2\phi f_p(\psi\partial_1\phi + \phi\partial_2\phi)) \\ &\quad +i(h\phi\partial_1 f_p + 2\phi f_p(\phi\partial_1\phi - \psi\partial_2\phi))\,)(\tilde{e}^{1235} - i\tilde{e}^{1236} - i\tilde{e}^{1245} - \tilde{e}^{1246})\,.\end{aligned} \tag{E.40}$$

From (E.11) we have

$$\phi(\phi^2+\psi^2)\partial_1 f_p + 2\phi f_p(\phi\partial_1\phi - \psi\partial_2\phi) + i(\,\phi(\phi^2+\psi^2)\partial_2 f_p + 2\phi f_p(\psi\partial_1\phi + \phi\partial_2\phi)\,) = 0 \tag{E.41}$$

and this is rewritten as follows:

$$\partial_1 f_{pR} - \partial_2 f_{pI} + f_{pR}\partial_1\log(\phi^2+\psi^2) - f_{pI}\partial_2\log(\phi^2+\psi^2) = 0\,, \tag{E.42}$$

$$\partial_1 f_{pI} + \partial_2 f_{pI} + f_{pI}\partial_1\log(\phi^2+\psi^2) + f_{pR}\partial_2\log(\phi^2+\psi^2) = 0\,, \tag{E.43}$$

where f_{pR} denotes a real part of f_p and f_{pI} denotes an imaginary part of f_p. We see that the functions,

$$f_{pR} = \frac{\phi}{\phi^2+\psi^2}\;,\;f_{pI} = \frac{\psi}{\phi^2+\psi^2} \tag{E.44}$$

are the solutions of the equations (E.42) and (E.43) and they satisfy $|f_p| = 1$. Therefore when $B_1 = B_2 = -\psi$, the deformed (3,0)-form $\tilde{\Upsilon}$ takes the following form

$$\tilde{\Upsilon} = i\frac{\phi + i\psi}{\phi^2+\psi^2}(\,\hat{e}^{135} + \tilde{e}^{245} + \tilde{e}^{236} - \tilde{e}^{146} + i(\tilde{e}^{235} - \tilde{e}^{145} - \tilde{e}^{136} - \tilde{e}^{246})\,)\,. \tag{E.45}$$

E.1 Bismut curvature and Hull curvature

Curvature 2-forms of connection $\tilde{\nabla}^{\pm}$ are defined by

$$\tilde{\mathcal{R}}^{\pm}_{ab} = d\tilde{\omega}^{\pm}_{ab} + \tilde{\omega}^{\pm}_{ab} \wedge \tilde{\omega}^{\pm}_{ab} = \frac{1}{2}\tilde{R}^{\pm}_{ab\mu\nu}\tilde{e}^{\mu\nu} \,, \tag{E.46}$$

where $\tilde{R}^{\pm}_{ab\mu\nu}$ $(a, b, \mu, \nu = 1, \cdots, 6)$ denotes curvature components. In what follows in this section, we use the notations

$$\Phi_1 = \partial_1 \log|\phi| \,,\ \Phi_2 = \partial_2 \log|\phi| \,,\ \Psi_1 = \partial_1^2 \log|\phi| \,,\ \Psi_2 = \partial_2^2 \log|\phi| \,,\ \Xi = \partial_2\partial_1 \log|\phi| \,. \tag{E.47}$$

The components of Bismut curvature is listed as follows:

$$\tilde{R}^{+}_{1212} = -\frac{2}{\phi^4}(\Psi_1 + \Psi_2) \,,\ \hat{R}^{+}_{1234} = 0 \,,\ \hat{R}^{+}_{1256} = 0 \,,$$

$$\tilde{R}^{+}_{1313} = \frac{-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1) + \phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2 + \psi^2)^2} \,,$$

$$\begin{aligned}\tilde{R}^{+}_{1314} \;=\; & -\frac{\psi}{\phi^4(\phi^2 + \psi^2)^2}(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1) \\ & +\phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1)) \,,\end{aligned}$$

$$\tilde{R}^{+}_{1323} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2 + \psi^2) + \psi^3(-3\Phi_1^2 + \Phi_2^2 - \Psi_2) - \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2)}{\phi^3(\phi^2 + \psi^2)^2} \,,$$

$$\tilde{R}^{+}_{1324} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2 + \psi^2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4(\phi^2 + \psi^2)^2} \,,$$

$$\tilde{R}^{+}_{1413} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2 + \psi^2) + \psi^3(-\Phi_1^2 + 3\Phi_2^2 + \Psi_1) + \phi^2\psi(-3\Phi_1^2 + 5\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2 + \psi^2)^2} \,,$$

$$\tilde{R}^{+}_{1414} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2 + \psi^2) + \phi^2\psi(3\Phi_1^2 - 5\Phi_2^2 - \Psi_1) + \psi^3(\Phi_1^2 - 3\Phi_2^2 - \Psi_1))}{\phi^4\left(\phi^2 + \psi^2\right)^2} \,,$$

$$\tilde{R}^{+}_{1423} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2)}{\phi^3\left(\phi^2 + \psi^2\right)^2} \,,$$

$$\tilde{R}^{+}_{1424} = -\frac{\psi(\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4(\phi^2 + \psi^2)^2} \,,$$

$$\tilde{R}^{+}_{1515} = \frac{-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1) + \phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2 + \psi^2)^2} \,,$$

$$\begin{aligned}\tilde{R}^{+}_{1516} \;=\; & -\frac{\psi}{\phi^4(\phi^2 + \psi^2)^2}(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(-\Phi_1^2 + \Phi_2^2 + \Psi_1) \\ & +\phi^3(-3\Phi_1^2 + 3\Phi_2^2 + \Psi_1)) \,,\end{aligned}$$

$$\tilde{R}^{+}_{1525} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) + \psi^3(-3\Phi_1^2+\Phi_2^2-\Psi_2) - \phi^2\psi(5\Phi_1^2-3\Phi_2^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{1526} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(5\Phi_1^2-3\Phi_2^2+\Psi_2) + \psi^3(3\Phi_1^2-\Phi_2^2+\Psi_2))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{1615} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) + \psi^3(-\Phi_1^2+3\Phi_2^2+\Psi_1) + \phi^2\psi(-3\Phi_1^2+5\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{1616} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(3\Phi_1^2-5\Phi_2^2-\Psi_1) + \psi^3(\Phi_1^2-3\Phi_2^2-\Psi_1))}{\phi^4\left(\phi^2+\psi^2\right)^2},$$

$$\tilde{R}^{+}_{1625} = \frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2) + (\Xi-4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2-3\Phi_2^2+\Psi_2) + \phi\psi^2(\Phi_1^2-\Phi_2^2+\Psi_2)}{\phi^3\left(\phi^2+\psi^2\right)^2},$$

$$\tilde{R}^{+}_{1626} = -\frac{\psi(\phi^2\psi(\Xi-8\Phi_1\Phi_2) + (\Xi-4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2-3\Phi_2^2+\Psi_2) + \phi\psi^2(\Phi_1^2-\Phi_2^2+\Psi_2))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{2313} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) + \psi^3(-\Phi_1^2+3\Phi_2^2+\Psi_1) + \phi^2\psi(-3\Phi_1^2+5\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{2314} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(3\Phi_1^2-5\Phi_2^2-\Psi_1) + \psi^3(\Phi_1^2-3\Phi_2^2+\Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{2323} = \frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2) + (\Xi-4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2-3\Phi_2^2+\Psi_2) + \phi\psi^2(\Phi_1^2-\Phi_2^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\begin{aligned}\tilde{R}^{+}_{2324} &= -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(\phi^2\psi(\Xi-8\Phi_1\Phi_2) + (\Xi-4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(\Phi_1^2-\Phi_2^2+\Psi_2) \\ &\quad +\phi^3(3\Phi_1^2-3\Phi_2^2+\Psi_2)),\end{aligned}$$

$$\tilde{R}^{+}_{2413} = \frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2) + (\Xi-4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2-3\Phi_2^2-\Psi_1) + \phi\psi^2(\Phi_1^2-\Phi_2^2-\Psi_2)}{\phi^3\left(\phi^2+\psi^2\right)^2},$$

$$\tilde{R}^{+}_{2414} = \frac{\psi(-\phi^2\psi(\Xi-8\Phi_1\Phi_2) - (\Xi-4\Phi_1\Phi_2)\psi^3 + \phi^3(-3\Phi_1^2+3\Phi_2^2+\Psi_1) + \phi\psi^2(-\Phi_1^2+\Phi_2^2+\Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{2423} = \frac{6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \psi^3(3\Phi_1^2-\Phi_2^2+\Psi_2) + \phi^2\psi(5\Phi_1^2-3\Phi_2^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{2424} = -\frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(5\Phi_1^2-3\Phi_2^2+\Psi_2) + \psi^3(3\Phi_1^2-\Phi_2^2+\Psi_2))}{\phi^4\left(\phi^2+\psi^2\right)^2},$$

$$\tilde{R}^{+}_{2515} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) + \psi^3(-\Phi_1^2+3\Phi_2^2+\Psi_1) + \phi^2\psi(-3\Phi_1^2+5\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{2516} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(3\Phi_1^2-5\Phi_2^2-\Psi_1) + \psi^3(\Phi_1^2-3\Phi_2^2+\Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{+}_{2525} = \frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2) + (\Xi-4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2-3\Phi_2^2+\Psi_2) + \phi\psi^2(\Phi_1^2-\Phi_2^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\begin{aligned}\tilde{R}^{+}_{2526} &= -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(\phi^2\psi(\Xi-8\Phi_1\Phi_2)+(\Xi-4\Phi_1\Phi_2)\psi^3+\phi\psi^2(\Phi_1^2-\Phi_2^2+\Psi_2)\\ &\quad +\phi^3(3\Phi_1^2-3\Phi_2^2+\Psi_2))\,,\end{aligned}$$

$$\tilde{R}^{+}_{2615} = \frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2)+(\Xi-4\Phi_1\Phi_2)\psi^3+\phi^3(3\Phi_1^2-3\Phi_2^2-\Psi_1)+\phi\psi^2(\Phi_1^2-\Phi_2^2-\Psi_2)}{\phi^3\left(\phi^2+\psi^2\right)^2}\,,$$

$$\tilde{R}^{+}_{2616} = \frac{\psi(-\phi^2\psi(\Xi-8\Phi_1\Phi_2)-(\Xi-4\Phi_1\Phi_2)\psi^3+\phi^3(-3\Phi_1^2+3\Phi_2^2+\Psi_1)+\phi\psi^2(-\Phi_1^2+\Phi_2^2+\Psi_1))}{\phi^4(\phi^2+\psi^2)^2}\,,$$

$$\tilde{R}^{+}_{2625} = \frac{6\phi^3\Phi_1\Phi_2-2\phi\psi^2\Phi_1\Phi_2-\Xi\phi(\phi^2+\psi^2)+\psi^3(3\Phi_1^2-\Phi_2^2+\Psi_2)+\phi^2\psi(5\Phi_1^2-3\Phi_2^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\tilde{R}^{+}_{2626} = -\frac{\psi(6\phi^3\Phi_1\Phi_2+2\phi\psi^2\Phi_1\Phi_2-\Xi\phi(\phi^2+\psi^2)+\phi^2\psi(5\Phi_1^2-3\Phi_2^2+\Psi_2)+\psi^3(3\Phi_1^2-\Phi_2^2+\Psi_2))}{\phi^4\left(\phi^2+\psi^2\right)^2}\,,$$

$$\tilde{R}^{+}_{3412} = \frac{\phi^2(\Phi_1^2+\Phi_2^2)-\psi^2(\Psi_1+\Psi_2)}{\phi^4(\phi^2+\psi^2)}\ ,\ \tilde{R}^{+}_{3434}=0\ ,\ \tilde{R}^{+}_{3456}=0\,,$$

$$\begin{aligned}\tilde{R}^{+}_{3535} &= -\frac{\Phi_1^2+\Phi_2^2}{\phi^2(\phi^2+\psi^2)}\ ,\ \tilde{R}^{+}_{3536}=\frac{(\Phi_1^2+\Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)}\,,\\ \tilde{R}^{+}_{3545} &= \frac{\psi(\Phi_1^2+\Phi_2^2)}{\phi^3(\phi^2+\psi^2)}\ ,\ \tilde{R}^{+}_{3546}=-\frac{(\Phi_1^2+\Phi_2^2)\psi^2}{\phi^4(\phi^2+\psi^2)}\,,\end{aligned}$$

$$\tilde{R}^{+}_{3635}=3\tilde{6}36=3\tilde{6}45=\tilde{R}^{+}_{3646}=0\,,\ \tilde{R}^{+}_{4535}=\tilde{R}^{+}_{4536}=\tilde{R}^{+}_{4545}=\tilde{R}^{+}_{4546}=0\,,$$

$$\begin{aligned}\tilde{R}^{+}_{4635} &= -\frac{\Phi_1^2+\Phi_2^2}{\phi^2(\phi^2+\psi^2)}\ ,\ \tilde{R}^{+}_{4636}=\frac{(\Phi_1^2+\Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)}\,,\\ \tilde{R}^{+}_{4645} &= \frac{\psi(\Phi_1^2+\Phi_2^2)}{\phi^3(\phi^2+\psi^2)}\ ,\ \tilde{R}^{+}_{4646}=-\frac{(\Phi_1^2+\Phi_2^2)\psi^2}{\phi^4(\phi^2+\psi^2)}\,,\end{aligned}$$

$$\tilde{R}^{+}_{5612} = \frac{\phi^2(\Phi_1^2+\Phi_2^2)-\psi^2(\Psi_1+\Psi_2)}{\phi^4(\phi^2+\psi^2)}\ ,\ \tilde{R}^{+}_{5634}=0\ ,\ 5\tilde{6}56=0\,.$$

Clearly, we can find the relations

$$-\tilde{R}^{+}_{12}+\tilde{R}^{+}_{34}+\tilde{R}^{+}_{56}=0\,,\ \tilde{R}^{+}_{13}=-\tilde{R}^{+}_{24}\,,\ \tilde{R}^{+}_{14}=\tilde{R}^{+}_{23}\,,\ \tilde{R}^{+}_{15}=-\tilde{R}^{+}_{26}\,,\ \tilde{R}^{+}_{16}=\tilde{R}^{+}_{25}\,. \tag{E.48}$$

Also, the components of Hull curvature are listed as follows:

$$\tilde{R}^{-}_{1212}=-\frac{2}{\phi^4}(\Psi_1+\Psi_2)\,,\ \tilde{R}^{-}_{1234}=\frac{(\Phi_1+\Phi_2)}{\phi^4}\,,\ \tilde{R}^{+}_{1256}=\frac{(\Phi_1+\Phi_2)}{\phi^4}\,,$$

$$\tilde{R}^{-}_{1313} = \frac{-\phi^2\psi(\Xi-8\Phi_1\Phi_2)-(\Xi-4\Phi_1\Phi_2)\psi^3+\phi\psi^2(-\Phi_1^2+\Phi_2^2+\Psi_1)+\phi^3(-3\Phi_1^2+3\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\tilde{R}^{-}_{1314} = \frac{-6\phi^3\Phi_1\Phi_2-2\phi\psi^2\Phi_1\Phi_2+\Xi\phi(\phi^2+\psi^2)+\psi^3(-\Phi_1^2+3\Phi_2^2+\Psi_1)+\phi^2\psi(-3\Phi_1^2+5\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2}\,,$$

$$\tilde{R}^{-}_{1323}=\frac{-6\phi^3\Phi_1\Phi_2-2\phi\psi^2\Phi_1\Phi_2+\Xi\phi(\phi^2+\psi^2)-\psi^3(2\Phi_1^2-2\Phi_2^2+\Psi_2)-\phi^2\psi(4\Phi_1^2-4\Phi_2^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1324}=\frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2)+(\Xi-4\Phi_1\Phi_2)\psi^3+\phi^3(4\Phi_1^2-2\Phi_2^2+\Psi_2)+\phi\psi^2(2\Phi_1^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\begin{aligned}\tilde{R}^{-}_{1413} &= -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(-\phi^2\psi(\Xi-8\Phi_1\Phi_2)-(\Xi-4\Phi_1\Phi_2)\psi^3+\phi\psi^2(-\Phi_1^2+\Phi_2^2+\Psi_1)\\ &\quad +\phi^3(-3\Phi_1^2+3\Phi_2^2+\Psi_1)),\end{aligned}$$

$$\tilde{R}^{-}_{1414}=\frac{\psi(6\phi^3\Phi_1\Phi_2+2\phi\psi^2\Phi_1\Phi_2-\Xi\phi(\phi^2+\psi^2)+\phi^2\psi(3\Phi_1^2-5\Phi_2^2-\Psi_1)+\psi^3(\Phi_1^2-3\Phi_2^2-\Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1423}=\frac{\psi(6\phi^3\Phi_1\Phi_2+2\phi\psi^2\Phi_1\Phi_2-\Xi\phi(\phi^2+\psi^2)+\phi^2\psi(4\Phi_1^2-4\Phi_2^2-\Psi_2)+\psi^3(2\Phi_1^2-2\Phi_2^2+\Psi_2))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1424}=-\frac{\psi(\phi^2\psi(\Xi-8\Phi_1\Phi_2)+(\Xi-4\Phi_1\Phi_2)\psi^3+\phi^3(4\Phi_1^2-2\Phi_2^2+\Psi_2)+\phi\psi^2(2\Phi_1^2+\Psi_2))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1515}=\frac{-\phi^2\psi(\Xi-8\Phi_1\Phi_2)-(\Xi-4\Phi_1\Phi_2)\psi^3+\phi\psi^2(-\Phi_1^2+\Phi_2^2+\Psi_1)+\phi^3(-3\Phi_1^2+3\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1516}=\frac{-6\phi^3\Phi_1\Phi_2-2\phi\psi^2\Phi_1\Phi_2+\Xi\phi(\phi^2+\psi^2)+\psi^3(-\Phi_1^2+3\Phi_2^2+\Psi_1)+\phi^2\psi(-3\Phi_1^2+5\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1525}=\frac{-6\phi^3\Phi_1\Phi_2-2\phi\psi^2\Phi_1\Phi_2+\Xi\phi(\phi^2+\psi^2)-\psi^3(2\Phi_1^2-2\Phi_2^2+\Psi_2)-\phi^2\psi(4\Phi_1^2-4\Phi_2^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1526}=\frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2)+(\Xi-4\Phi_1\Phi_2)\psi^3+\phi^3(4\Phi_1^2-2\Phi_2^2+\Psi_2)+\phi\psi^2(2\Phi_1^2+\Psi_2)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\begin{aligned}\tilde{R}^{-}_{1615} &= -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(-\phi^2\psi(\Xi-8\Phi_1\Phi_2)-(\Xi-4\Phi_1\Phi_2)\psi^3+\phi\psi^2(-\Phi_1^2+\Phi_2^2+\Psi_1)\\ &\quad +\phi^3(-3\Phi_1^2+3\Phi_2^2+\Psi_1)),\end{aligned}$$

$$\tilde{R}^{-}_{1616}=\frac{\psi(6\phi^3\Phi_1\Phi_2+2\phi\psi^2\Phi_1\Phi_2-\Xi\phi(\phi^2+\psi^2)+\phi^2\psi(3\Phi_1^2-5\Phi_2^2-\Psi_1)+\psi^3(\Phi_1^2-3\Phi_2^2-\Psi_1))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1625}=\frac{\psi(6\phi^3\Phi_1\Phi_2+2\phi\psi^2\Phi_1\Phi_2-\Xi\phi(\phi^2+\psi^2)+\phi^2\psi(4\Phi_1^2-4\Phi_2^2-\Psi_2)+\psi^3(2\Phi_1^2-2\Phi_2^2+\Psi_2))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{1626}=-\frac{\psi(\phi^2\psi(\Xi-8\Phi_1\Phi_2)+(\Xi-4\Phi_1\Phi_2)\psi^3+\phi^3(4\Phi_1^2-2\Phi_2^2+\Psi_2)+\phi\psi^2(2\Phi_1^2+\Psi_2))}{\phi^4(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{2313}=\frac{-6\phi^3\Phi_1\Phi_2-2\phi\psi^2\Phi_1\Phi_2+\Xi\phi(\phi^2+\psi^2)+\psi^3(-2\Phi_1^2+2\Phi_2^2+\Psi_1)+\phi^2\psi(-4\Phi_1^2+4\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{2314}=\frac{\phi^2\psi(\Xi-8\Phi_1\Phi_2)+(\Xi-4\Phi_1\Phi_2)\psi^3+\phi^3(2\Phi_1^2-4\Phi_2^2-\Psi_1)-\phi\psi^2(2\Phi_2^2+\Psi_1)}{\phi^3(\phi^2+\psi^2)^2},$$

$$\tilde{R}^{-}_{2323} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{-}_{2324} = \frac{6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{-}_{2413} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(4\Phi_1^2 - 4\Phi_2^2 - \Psi_1) + \psi^3(2\Phi_1^2 - 2\Phi_2^2 - \Psi_1))}{\phi^4(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{-}_{2414} = \frac{\psi(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(-2\Phi_1^2 + 4\Phi_2^2 + \Psi_1) + \phi\psi^2(2\Phi_2^2 + \Psi_1))}{\phi^4(\phi^2+\psi^2)^2} ,$$

$$\begin{aligned} \tilde{R}^{+}_{2423} \quad = \quad & -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2) \\ & +\phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2)) , \end{aligned}$$

$$\tilde{R}^{-}_{2424} = -\frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4\left(\phi^2+\psi^2\right)^2} ,$$

$$\tilde{R}^{-}_{2515} = \frac{-6\phi^3\Phi_1\Phi_2 - 2\phi\psi^2\Phi_1\Phi_2 + \Xi\phi(\phi^2+\psi^2) + \psi^3(-2\Phi_1^2 + 2\Phi_2^2 + \Psi_1) + \phi^2\psi(-4\Phi_1^2 + 4\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{-}_{2516} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(2\Phi_1^2 - 4\Phi_2^2 - \Psi_1) - \phi\psi^2(2\Phi_2^2 + \Psi_1)}{\phi^3(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{+}_{2525} = \frac{\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{+}_{2526} = \frac{6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2)}{\phi^3(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{-}_{2615} = \frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(4\Phi_1^2 - 4\Phi_2^2 - \Psi_1) + \psi^3(2\Phi_1^2 - 2\Phi_2^2 - \Psi_1))}{\phi^4(\phi^2+\psi^2)^2} ,$$

$$\tilde{R}^{-}_{2616} = \frac{\psi(-\phi^2\psi(\Xi - 8\Phi_1\Phi_2) - (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi^3(-2\Phi_1^2 + 4\Phi_2^2 + \Psi_1) + \phi\psi^2(2\Phi_2^2 + \Psi_1))}{\phi^4(\phi^2+\psi^2)^2} ,$$

$$\begin{aligned} \tilde{R}^{-}_{2625} \quad = \quad & -\frac{\psi}{\phi^4(\phi^2+\psi^2)^2}(\phi^2\psi(\Xi - 8\Phi_1\Phi_2) + (\Xi - 4\Phi_1\Phi_2)\psi^3 + \phi\psi^2(\Phi_1^2 - \Phi_2^2 + \Psi_2) \\ & +\phi^3(3\Phi_1^2 - 3\Phi_2^2 + \Psi_2)) , \end{aligned}$$

$$\tilde{R}^{-}_{2626} = -\frac{\psi(6\phi^3\Phi_1\Phi_2 + 2\phi\psi^2\Phi_1\Phi_2 - \Xi\phi(\phi^2+\psi^2) + \phi^2\psi(5\Phi_1^2 - 3\Phi_2^2 + \Psi_2) + \psi^3(3\Phi_1^2 - \Phi_2^2 + \Psi_2))}{\phi^4\left(\phi^2+\psi^2\right)^2} ,$$

$$\tilde{R}^{-}_{3412} = \tilde{R}^{-}_{3434} = \tilde{R}^{-}_{3456} = 0 ,$$

$$\tilde{R}^{-}_{3535} = -\frac{\Phi_1^2 + \Phi_2^2}{\phi^4 + \phi^2\psi^2} , \ \hat{R}^{-}_{3536} = \hat{R}^{-}_{3545} = 0 , \ \hat{R}^{-}_{3546} = -\frac{(\Phi_1^2 + \Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)} ,$$

$$\hat{R}^-_{3635} = -\frac{(\Phi_1^2+\Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)}\,,\ \hat{R}^-_{3636} = \hat{R}^-_{3645} = 0\,,\ \hat{R}^-_{3646} = \frac{(\Phi_1^2+\Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)}\,,$$
$$\hat{R}^-_{4535} = \frac{(\Phi_1^2+\Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)}\,,\ \hat{R}^-_{4536} = \hat{R}^-_{4545} = 0\,,\ \hat{R}^-_{4546} = \frac{(\Phi_1^2+\Phi_2^2)\psi}{\phi^3(\phi^2+\psi^2)}\,,$$
$$\tilde{R}^-_{4635} = -\frac{(\Phi_1^2+\Phi_2^2)\psi^2}{\phi^4(\phi^2+\psi^2)}\,,\ \tilde{R}^-_{4636} = \tilde{R}^-_{4645} = 0\,,\ \tilde{R}^-_{4646} = -\frac{(\Phi_1^2+\Phi_2^2)\psi^2}{\phi^4(\phi^2+\psi^2)}\,,$$
$$\tilde{R}^-_{5612} = \tilde{R}^-_{5634} = \tilde{R}^-_{5656} = 0\,.$$

These components of the Bismut and the Hull curvature satisfy the identity $\tilde{R}^+_{ab\mu\nu} - \tilde{R}^-_{\mu\nu ab} = d\tilde{T}_{ab\mu\nu} = 0$.

Appendix F

General Wahlquist metrics in all dimensions

As was discussed previous chapters, totally skew symmetric torsion tensors appear in various class of G-structures naturally. In case of related physics, the torsion tensors play an important role. Apart from G-structure, totally skew symmetric torsion tensors also contribute to generalizing Killing-Yano symmetry. Killing-Yano symmetry is significant for integrability on higher-dimensional rotating black hole spacetimes with spherical horizon topology [58, 59, 60, 61]. In this appendix, we give an outline of the general Wahlquist solution. This appendix is based on Ref. [3].

The Wahlquist metric [62, 63, 64] is a stationary, axially symmetric perfect fluid solution with the state equation $\rho + 3p = const$. The Wahlquist space time has some geometric properties of the Kerr spacetime, which is obtained by limiting case of the Wahlquist spacetime [62, 65]. The kerr metric is the only vacuum solution admitting rank-2 Killing-Yano tensor [66]. It is known that the rank-2 Killing-Yano tensor generates two Killing vector fields and Killing-Stäkel tensor in the Kerr spacetime. This implies that existence of the rank-2 Killing-Yano tensor characterize geometric properties of the Kerr spacetime. However, Killing-Yano symmetry of the Wahlquist spacetime is still unclear. In the following of this appendix, we firstly find a rank-2 generalized closed conformal Killing-Yano (GCCKY) tensor with torsion [57]. Secondly, we attempt to construct a new family of rotating perfect fluid solutions which generalize the Wahlquist metric to higher dimension.

F.1 Killing-Yano symmetry of the Wahlquist spacetime

The Wahlquist metric in 4 dimensions [62, 63, 64] can be written in a local coordinate system (z, w, τ, σ) as

$$g = (v_1 + v_2)\left(\frac{dz^2}{U} + \frac{dw^2}{V}\right) + \frac{U}{v_1 + v_2}(d\tau + v_2 d\sigma)^2 - \frac{V}{v_1 + v_2}(d\tau - v_1 d\sigma)^2\,, \tag{F.1}$$

where

$$U = Q_0 + a_1 \frac{\sinh(2\beta z)}{2\beta} - \nu_0 \frac{\cosh(2\beta z) - 1}{2\beta^2} - \frac{\mu_0}{\beta^2}\left(\frac{\cosh(2\beta z) - 1}{2\beta^2} - \frac{z\sinh(2\beta z)}{2\beta}\right),$$
$$V = Q_0 + a_2 \frac{\sin(2\beta w)}{2\beta} + \nu_0 \frac{1 - \cos(2\beta w)}{2\beta^2} + \frac{\mu_0}{\beta^2}\left(\frac{1 - \cos(2\beta w)}{2\beta^2} - \frac{w\sin(2\beta w)}{2\beta}\right), \quad \text{(F.2)}$$

and

$$v_1 = \frac{\cosh(2\beta z) - 1}{2\beta^2}, \; v_2 = \frac{1 - \cos(2\beta w)}{2\beta^2}. \quad \text{(F.3)}$$

The metric contains six real constants Q_0, a_1, a_2, ν_0, μ_0 and β. As was shown in [62], one can take the limit $\beta \to 0$. In the limit, the metric reduces to the Kerr-NUT-(A)dS metric [67].

The Wahlquist metric provides the stress-energy tensor for perfect fluids of the energy density ρ, pressure p and 4-velocity u with $u_\mu u^\mu = -1$, which is written as

$$T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}. \quad \text{(F.4)}$$

The 4-velocity is given by

$$u^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{-g_{\tau\tau}}}\frac{\partial}{\partial \tau}, \quad \text{(F.5)}$$

where $g_{\tau\tau} = (U - V)/(v_1 + v_2)$. If u lies on the 2-plane spanned by two Killing vector fields ∂_τ and ∂_σ, then u can be written as $u = N(\partial_\tau + \Omega\partial_\sigma)$, where N and Ω are functions in general. In particular, when Ω is constant, the perfect fluid is said to be rigidly rotating. Namely, the Wahlquist solution represents rigidly rotating perfect fluids. The energy density and pressure are given by

$$\rho = -\mu_0 - 3\beta^2 g_{\tau\tau}, \; p = \mu_0 + \beta^2 g_{\tau\tau}. \quad \text{(F.6)}$$

Thus, the equation of state is $\rho + 3p = 2\mu_0$.

F.1.1 Killing-Yano symmetry

From now on, we demonstrate that the Wahlquist metric (F.1) admits a rank-2 GCCKY tensor. For the purpose, we introduce the coordinates x and y defined by

$$x^2 = v_1, \; y^2 = v_2, \quad \text{(F.7)}$$

and hence

$$dz^2 = \frac{dx^2}{\beta^2 x^2 + 1}, \qquad dw^2 = \frac{dy^2}{1 - \beta^2 y^2}. \quad \text{(F.8)}$$

The metric is then written as

$$g = \frac{x^2 + y^2}{U(1 + \beta^2 x^2)}dx^2 + \frac{x^2 + y^2}{V(1 - \beta^2 y^2)}dy^2 + \frac{U}{x^2 + y^2}(d\tau + y^2 d\sigma)^2 - \frac{V}{x^2 + y^2}(d\tau - x^2 d\sigma)^2$$

with the functions

$$
\begin{aligned}
U &= Q_0 + a_1 x\sqrt{1+\beta^2x^2} - \nu_0 x^2 - \frac{\mu_0}{\beta^2}\left(x^2 - \frac{x\text{Arcsinh}(\beta x)\sqrt{1+\beta^2x^2}}{\beta}\right) , \\
V &= Q_0 + a_2 y\sqrt{1-\beta^2y^2} + \nu_0 y^2 + \frac{\mu_0}{\beta^2}\left(y^2 - \frac{y\text{Arcsin}(\beta y)\sqrt{1-\beta^2y^2}}{\beta}\right) .
\end{aligned}
$$

Furthermore, taking the Wick rotation $y \to \sqrt{-1}y$ (with $a_2 \to -\sqrt{-1}a_2$ to keep the metric function V real) and changing the sign $\sigma \to -\sigma$, we obtain the Euclideanised expression for the Wahlquist metric

$$
g_E = \frac{f_1(x^2-y^2)}{\Xi_1}dx^2 + \frac{f_2(y^2-x^2)}{\Xi_2}dy^2 + \frac{\Xi_1}{x^2-y^2}(d\tau + y^2 d\sigma)^2 + \frac{\Xi_2}{y^2-x^2}(d\tau + x^2 d\sigma)^2 , \quad \text{(F.9)}
$$

where

$$
\begin{aligned}
f_1 &= \frac{1}{\sqrt{1+\beta^2x^2}} , \; f_2 = \frac{1}{\sqrt{1+\beta^2y^2}} , \\
\Xi_1 &= Q_0 + a_1 x\sqrt{1+\beta^2x^2} - \nu_0 x^2 - \frac{\mu_0}{\beta^2}\left(x^2 - \frac{x\text{Arcsinh}(\beta x)\sqrt{1+\beta^2x^2}}{\beta}\right) , \\
\Xi_2 &= Q_0 + a_2 y\sqrt{1+\beta^2y^2} - \nu_0 y^2 - \frac{\mu_0}{\beta^2}\left(y^2 - \frac{y\text{Arcsinh}(\beta y)\sqrt{1+\beta^2y^2}}{\beta}\right) .
\end{aligned}
$$

The form of the metric (F.9) is symmetric with respect to the coordinates (x, y) and precisely fits into type A of the classification in [68], that is, the Wahlquist spacetime admits a rank-2 GCCKY tensor. In fact, if we introduce an orthonormal frame

$$
\begin{aligned}
e^1 &= f_1\sqrt{\frac{x^2-y^2}{\Xi_1}}dx , \; e^2 = f_2\sqrt{\frac{y^2-x^2}{\Xi_2}}dy , \\
e^{\hat{1}} &= \sqrt{\frac{\Xi_1}{x^2-y^2}}(d\tau + y^2 d\sigma) , \; e^{\hat{2}} = \sqrt{\frac{\Xi_2}{y^2-x^2}}(d\tau + x^2 d\sigma) ,
\end{aligned}
$$

the rank-2 GCCKY tensor is given by

$$
h = x\, e^1 \wedge e^{\hat{1}} + y\, e^2 \wedge e^{\hat{2}} \quad \text{(F.10)}
$$

with the skew-symmetric torsion

$$
T = \frac{2x(f_1-f_2)}{f_1 f_2(x^2-y^2)}\sqrt{\frac{\Xi_2}{y^2-x^2}}e^1 \wedge e^{\hat{1}} \wedge e^{\hat{2}} + \frac{2y(f_2-f_1)}{f_1 f_2(y^2-x^2)}\sqrt{\frac{\Xi_1}{x^2-y^2}}e^2 \wedge e^{\hat{2}} \wedge e^{\hat{1}} . \quad \text{(F.11)}
$$

The torsion vanishes when we take the limit $\beta \to 0$. This suggests that the torsion is related to the perfect fluid. However, the physical meaning of the torsion is unclear.

F.2 Higher-dimensional Wahlquist spacetimes

We have seen that the Wahlquist metric (F.9) admits a rank-2 GCCKY tensor and its Euclidean form precisely fits into type A of the classification [68]. It seems to be reasonable to consider a higher-dimensional generalisation of the Wahlquist metric. In this section, we attempt to solve Einstein equations for perfect fluids in higher dimensions by employing type A metrics obtained in [68] as an ansatz.

Hereafter, we slightly change our notation. To treat higher-dimensional metrics in both even and odd dimensions, we introduce ε, where $\varepsilon = 0$ for even and $\varepsilon = 1$ for odd dimensions. The dimension is denoted by $D = 2n + \varepsilon$. The Latin indices $a, b, \cdots$ run from 1 to D and the Greece indices $\mu, \nu, \cdots$ run from 1 to n. In the notation, we consider the following form of metrics admitting a rank-2 GCCKY tensor in D dimensions:

$$g^{(D)} = \sum_{\mu=1}^{n} \frac{f_\mu^2}{P_\mu} dx_\mu^2 + \sum_{\mu=1}^{n} P_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 + \varepsilon S \left(\sum_{k=0}^{n} A^{(k)} d\psi_k \right)^2 , \tag{F.12}$$

where

$$P_\mu = \frac{\Xi_\mu}{U_\mu} \,,\; U_\mu = \prod_{\nu \neq \mu} (x_\mu^2 - x_\nu^2) \,,\; S = \frac{k^2}{A^{(n)}} \,,\; f_\mu = \frac{1}{\sqrt{1 + \beta^2 x_\mu^2}} \,, \tag{F.13}$$

and $A_\mu^{(k)}$ $(k = 0, \cdots, n-1)$ and $A^{(k)}$ $(k = 0, \cdots, n)$ are the k-th elementary symmetric functions in $(x_1^2, x_2^2, \cdots x_n^2)$ defined by

$$\sum_{k=0}^{n-1} A_\mu^{(k)} t^k = \prod_{\nu \neq \mu} (1 + t x_\nu^2) \,,\; \sum_{k=0}^{n} A^{(k)} t^k = \prod_{\nu=1}^{n} (1 + t x_\nu^2). \tag{F.14}$$

The metric contains n unknown functions $\Xi_\mu(x_\mu)$ which depend only on single valuables x_μ, and a constant k^2.

F.2.1 Even dimensions

We determine the functions Ξ_μ from the Einstein equation for perfect fluids in even dimensions. The metric ansatz in $2n$ dimensions is given by

$$g^{(2n)} = \sum_{\mu=1}^{n} \frac{f_\mu^2}{P_\mu} dx_\mu^2 + \sum_{\mu=1}^{n} P_\mu \left(\sum_{k=0}^{n-1} A_\mu^{(k)} d\psi_k \right)^2 . \tag{F.15}$$

For the metric, the off-diagonal components of the Ricci curvature are

$$R_{\mu\hat{\mu}} = R_{\mu\nu} = R_{\mu\hat{\nu}} = 0 \,,\, R_{\hat{\mu}\hat{\nu}} = \beta^2 (D-2) \sqrt{P_\mu} \sqrt{P_\nu} \,. \tag{F.16}$$

The diagonal components are

$$R_{\mu\mu} = I_\mu(P_T) + \beta^2 \left(I_\mu(P_T^{(2)}) + \frac{3}{2} x_\mu \partial_\mu P_T + P_T \right) \,,\, R_{\hat{\mu}\hat{\mu}} = R_{\mu\mu} + \beta^2 (D-2) P_\mu \,, \tag{F.17}$$

where

$$P_T = \sum_{\mu=1}^{n} P_\mu \,,\; P_T^{(2)} = \sum_{\mu=1}^{n} x_\mu^2 P_\mu \tag{F.18}$$

and I_μ are differential operators given by

$$I_\mu = -\frac{1}{2}\frac{\partial^2}{\partial x_\mu^2} + \frac{1}{x_\mu^2 - x_\nu^2}\left(x_\mu \frac{\partial}{\partial x_\mu} - x_\nu \frac{\partial}{\partial x_\nu}\right) . \tag{F.19}$$

It should be emphasized that our metric ansatz is now expressed with Euclidean signature, so that we have to consider the Euclideanised Einstein equation for perfect fluids,

$$R_{ab} - \frac{1}{2}R g_{ab} = -(\rho + p)u_a u_b + p g_{ab} \,, \tag{F.20}$$

where $u^a u_a = 1$. Eliminating the scalar curvature, we obtain the equation in a more convenient form

$$R_{ab} = -(\rho + p)u_a u_b + \frac{\rho - p}{D-2} g_{ab} \,. \tag{F.21}$$

Moreover, to solve the equation, we assume that perfect fluids are rigidly rotating, namely, the velocity u is written as $u = N\partial_{\psi_0}$ where N is the normalization function. Since we have $N = 1/\sqrt{P_T}$ from $u^a u_a = 1$, the velocity is given in the canonical frame as

$$u = \frac{1}{\sqrt{P_T}} \sum_{\mu=1}^{n} \sqrt{P_\mu} e_{\hat{\mu}} \,. \tag{F.22}$$

Under the assumption, together with (F.16) and (F.17), the Einstein equation to solve reduces to

$$\frac{\rho - p}{D-2} = R_{11} = R_{22} = \cdots = R_{nn} \,, \tag{F.23}$$

$$\frac{\rho + p}{D-2} = -\beta^2 P_T \,. \tag{F.24}$$

To solve the equations (F.23), we need to notice that the $\mu\mu$-components of the Ricci curvature, $R_{\mu\mu}$, can be written in a simple form. Calculating $R_{\mu\mu}$ in terms of Ξ_μ and its derivatives, we obtain

$$R_{\mu\mu} = -\frac{1}{2x_\mu}\left(\frac{G_\mu}{U_\mu} - \sum_{\nu\neq\mu} \frac{2x_\mu}{x_\mu^2 - x_\nu^2}\left(\frac{F_\mu}{U_\mu} + \frac{F_\nu}{U_\nu}\right)\right) - 2\beta^2 P_T \,, \tag{F.25}$$

where

$$G_\mu = x_\mu(1 + \beta^2 x_\mu^2)\Xi_\mu'' + \beta^2 x_\mu^2 \Xi_\mu' - 4\beta^2 x_\mu \Xi_\mu \,, \tag{F.26}$$

$$F_\mu = x_\mu(1 + \beta^2 x_\mu^2)\Xi_\mu' - (1 + 2\beta^2 x_\mu^2)\Xi_\mu \,. \tag{F.27}$$

Noticing that $G_\mu = F'_\mu$ and that

$$\frac{\partial F_T}{\partial x_\mu} = \frac{F'_\mu}{U_\mu} - \sum_{\nu\neq\mu} \frac{2x_\mu}{x_\mu^2 - x_\nu^2}\left(\frac{F_\mu}{U_\mu} + \frac{F_\nu}{U_\nu}\right), \tag{F.28}$$

where

$$F_T = \sum_{\rho=1}^{n} \frac{F_\rho}{U_\rho}, \tag{F.29}$$

we obtain the following expressions for $\mu\mu$-components of the Ricci curvature:

$$R_{\mu\mu} = -\frac{1}{2x_\mu}\frac{\partial F_T}{\partial x_\mu} - 2\beta^2 P_T. \tag{F.30}$$

From (F.30), $R_{\mu\mu} - R_{\nu\nu} = 0$ implies that

$$\left[\frac{1}{x_\mu}\frac{\partial}{\partial x_\mu} - \frac{1}{x_\nu}\frac{\partial}{\partial x_\nu}\right] F_T = 0. \tag{F.31}$$

This can be solved by $F_T = F_T(\xi)$, where $F_T(\xi)$ is an arbitrary function of $\xi = \sum_{\mu=1}^n x_\mu^2$. Substituting it into (F.29) and differentiating by $\partial_{x_1}\partial_{x_2}\cdots\partial_{x_n}$ the both sides of the equation multiplied by the factor $\prod_{\mu\neq\nu}(x_\mu^2 - x_\nu^2)$, we arrive at the condition $F_T^{(n)}(\xi) = 0$, which implies that $F_T(\xi)$ is an n-th order polynomial in ξ. Furthermore, going back to (F.29) again and comparing the coefficients of the equation, we find that F_T must be a linear function. Namely, to be consistent with (F.29), the function must be chosen as $F_T(\xi) = C_1\xi + C_2$ where C_1 and C_2 are constants. Then, using the identities

$$\sum_{\mu=1}^{n} \frac{x_\mu^{2j}}{U_\mu} = 0 \quad (j = 0, \cdots, n-2), \sum_{\mu=1}^{n} \frac{x_\mu^{2(n-1)}}{U_\mu} = 1, \sum_{\mu=1}^{n} \frac{x_\mu^{2n}}{U_\mu} = \sum_{\mu=1}^{n} x_\mu^2, \tag{F.32}$$

we obtain

$$F_\mu = \sum_{k=0}^{n} c_{2k} x_\mu^{2k}, \tag{F.33}$$

where c_{2k} $(k = 0, 1, \cdots, n)$ are constants with $C_1 = c_{2n}$ and $C_2 = c_{2(n-1)}$. In the end, using (F.27) and (F.33), the present problem of solving the Einstein equation (F.20) has been reduced to that of solving first-order ordinary differential equations for Ξ_μ:

$$\Xi'_\mu - \frac{1 + 2\beta^2 x_\mu^2}{x_\mu(1+\beta^2 x_\mu^2)}\Xi_\mu - \frac{\sum_{k=0}^n c_{2k}x_\mu^{2k}}{x_\mu(1+\beta^2 x_\mu^2)} = 0 \tag{F.34}$$

The general solution is

$$\Xi_\mu = \sum_{k=0}^{n} c_{2k}\phi_{2k}(x_\mu) + a_\mu x_\mu\sqrt{1+\beta^2 x_\mu^2}, \tag{F.35}$$

where a_μ are integral constants, $\phi_{2k}(x)$ for $k \geq 1$ are given by

$$\phi_{2k}(x) = x\sqrt{1+\beta^2x^2}\int_0^x \frac{t^{2(k-1)}dt}{(1+\beta^2t^2)^{3/2}} \tag{F.36}$$

and $\phi_0(x) \equiv -1$. The solution in $2n$ dimensions contains parameters c_{2k} ($k = 0, \cdots, n$), a_μ ($\mu = 1, \cdots, n$) and β. Note that for low k, we have

$$\begin{aligned}
\phi_2(x) &= x^2 \ , \ \phi_4(x) = -\frac{x}{\beta^2}\left(x - \frac{\mathrm{Arcsinh}(\beta x)\sqrt{1+\beta^2x^2}}{\beta}\right) , \\
\phi_6(x) &= \frac{3x}{2\beta^4}\left(x + \frac{\beta^2}{3}x^3 - \frac{\mathrm{Arcsinh}(\beta x)\sqrt{1+\beta^2x^2}}{\beta}\right) , \\
\phi_8(x) &= -\frac{15x}{8\beta^6}\left(x + \frac{\beta^2}{3}x^3 - \frac{2\beta^4}{15}x^5 - \frac{\mathrm{Arcsinh}(\beta x)\sqrt{1+\beta^2x^2}}{\beta}\right) .
\end{aligned} \tag{F.37}$$

In 4 dimensions, for $\mu = 1, 2$, we obtain

$$\Xi_\mu = -c_0 + c_2x_\mu^2 + a_\mu x_\mu\sqrt{1+\beta^2x_\mu^2} - \frac{c_4x_\mu}{\beta^2}\left(x_\mu - \frac{\mathrm{Arcsinh}(\beta x_\mu)\sqrt{1+\beta^2x_\mu^2}}{\beta}\right) . \tag{F.38}$$

The form coincides with the Wahlquist solution.

In the limit $\beta \to 0$, we have $\phi_{2k} \to x^{2k}/(2k-1)$. That is,

$$\Xi_\mu = \sum_{k=0}^{n} \tilde{c}_{2k}x_\mu^{2k} + a_\mu x_\mu \, , \tag{F.39}$$

where $\tilde{c}_{2k} = c_{2k}/(2k-1)$. This takes the same form as Kerr-NUT-(A)dS metrics in $2n$ dimensions found by Chen-Lü-Pope [61].

Finally, let us comment about the equation of state. From (F.23), (F.24), (F.30) and (F.33), we have

$$\frac{2\rho}{D-2} = -c_{2n} - 3\beta^2P_T \, , \quad \frac{2p}{D-2} = c_{2n} + \beta^2P_T \, . \tag{F.40}$$

Hence, the equation of state is $\rho + 3p = (D-2)c_{2n}$.

F.2.2 Odd dimensions

Let us consider odd dimensions $D = 2n+1$. The metric ansatz in $2n+1$ dimensions is given by

$$g^{(2n+1)} = \sum_{\mu=1}^{n} \frac{f_\mu^2}{P_\mu}dx_\mu^2 + \sum_{\mu=1}^{n} P_\mu\left(\sum_{k=0}^{n-1} A_\mu^{(k)}d\psi_k\right)^2 + S\left(\sum_{k=0}^{n} A^{(k)}d\psi_k\right)^2 \tag{F.41}$$

with unknown functions Ξ_μ. The off-diagonal components of the Ricci curvature are

$$R_{\mu\nu} = R_{\mu\hat{\nu}} = R_{\mu\hat{\mu}} = R_{\mu 0} = 0\,,\; R_{\hat{\mu}\hat{\nu}} = \beta^2(D-2)\sqrt{P_\mu}\sqrt{P_\nu}\,,\; R_{\hat{\mu}0} = \beta^2(D-2)\sqrt{P_\mu}\sqrt{S}\,. \quad \text{(F.42)}$$

The diagonal components are

$$\begin{aligned} R_{\mu\mu} &= I_\mu(\tilde{P}_T) + \beta^2\left(I_\mu(\tilde{P}_T^{(2)}) + \frac{3}{2}x_\mu\partial_\mu\tilde{P}_T + \tilde{P}_T\right) - \frac{1}{2x_\mu}\frac{\partial}{\partial x_\mu}\left(\tilde{P}_T + \beta^2\tilde{P}_T^{(2)}\right)\,, \\ R_{\hat{\mu}\hat{\mu}} &= R_{\mu\mu} + \beta^2(D-2)P_\mu\,, \\ R_{00} &= -\sum_{\mu=1}^{n}\frac{1}{x_\mu}\frac{\partial}{\partial x_\mu}\left(\tilde{P}_T + \beta^2\tilde{P}_T^{(2)}\right) + \beta^2\tilde{P}_T + \beta^2(D-2)S\,, \end{aligned} \quad \text{(F.43)}$$

where

$$\tilde{P}_T = \sum_{\mu=1}^{n}\tilde{P}_\mu = P_T + S \;\;,\;\; \tilde{P}_T^{(2)} = \sum_{\mu=1}^{n}x_\mu^2\tilde{P}_\mu = P_T^{(2)} \quad \text{(F.44)}$$

and

$$\tilde{P}_\mu = \frac{\tilde{\Xi}_\mu}{U_\mu} \;\;,\;\; \tilde{\Xi}_\mu = \Xi_\mu - (-1)^n\frac{k^2}{x_\mu^2}. \quad \text{(F.45)}$$

We assume that the velocity u lies in the plane of the Killing vectors

$$u = \frac{1}{\sqrt{\tilde{P}_T}}\left(\sum_{\mu=1}^{n}\sqrt{P_\mu}e_{\hat{\mu}} + \sqrt{S}e_0\right)\,. \quad \text{(F.46)}$$

The equation reduces to

$$\frac{\rho - p}{D-2} = R_{11} = R_{22} = \cdots = R_{nn}\,, \quad \text{(F.47)}$$

$$\frac{\rho + p}{D-2} = -\beta^2\tilde{P}_T\,, \quad \text{(F.48)}$$

and for all μ,

$$R_{00} = R_{\mu\mu} + \beta^2(D-2)S\,. \quad \text{(F.49)}$$

Similarly to even dimensions, we find from the direct calculation that the $\mu\mu$- and 00-components of the Ricci curvature can be written in the simple form

$$R_{\mu\mu} = -\frac{1}{2x_\mu}\frac{\partial\tilde{F}_T}{\partial x_\mu} - 2\beta^2\tilde{P}_T\,, \quad \text{(F.50)}$$

$$R_{00} = -\sum_{\mu=1}^{n}\frac{\tilde{F}_\mu}{x_\mu^2 U_\mu} - 2\beta^2\tilde{P}_T + \beta^2(D-2)S\,, \quad \text{(F.51)}$$

where

$$\tilde{F}_T = \sum_{\mu=1}^{n}\frac{\tilde{F}_\mu}{U_\mu} \quad \text{(F.52)}$$

and

$$\tilde{F}_\mu = x_\mu(1+\beta^2 x_\mu^2)\tilde{\Xi}'_\mu - \beta^2 x_\mu^2 \tilde{\Xi}_\mu\,. \tag{F.53}$$

As was discussed in even dimensions (cf. (F.33)), the equation (F.47) requires that $\tilde{F}_\mu$ are given as

$$\tilde{F}_\mu = \sum_{k=0}^{n} c_{2k} x_\mu^{2k}\,. \tag{F.54}$$

Indeed, by virtue of (F.50) and (F.51), we easily see that (F.54) together with $c_0 = 0$ solves (F.47) and (F.49). From the equality of (F.53) and (F.54), we obtain the first-order ordinary differential equations

$$\tilde{\Xi}'_\mu - \frac{\beta^2 x_\mu}{1+\beta^2 x_\mu^2}\tilde{\Xi}_\mu - \frac{\sum_{k=1}^{n} c_{2k} x_\mu^{2k-1}}{1+\beta^2 x_\mu^2} = 0\,. \tag{F.55}$$

The general solution is

$$\tilde{\Xi}_\mu = \sum_{k=1}^{n} c_{2k}\tilde{\phi}_{2k}(x_\mu) + a_\mu\sqrt{1+\beta^2 x_\mu^2}\,, \tag{F.56}$$

where a_μ are integral constants and

$$\tilde{\phi}_{2k}(x) = \sqrt{1+\beta^2 x^2}\int_0^x \frac{t^{2k-1}dt}{(1+\beta^2 t^2)^{3/2}}\,. \tag{F.57}$$

Note that for low k, $\tilde{\phi}_{2k}$ $(k = 1, 2, \cdots)$ are written as

$$\begin{aligned}
\tilde{\phi}_2(x) &= -\frac{1}{\beta^2}\left(1-\sqrt{1+\beta^2 x^2}\right),\ \tilde{\phi}_4(x) = \frac{2}{\beta^4}\left(1+\frac{\beta^2}{2}x^2-\sqrt{1+\beta^2 x^2}\right),\\
\tilde{\phi}_6(x) &= -\frac{8}{3\beta^6}\left(1+\frac{\beta^2}{2}x^2-\frac{\beta^4}{8}x^4-\sqrt{1+\beta^2 x^2}\right),\\
\tilde{\phi}_8(x) &= \frac{16}{5\beta^8}\left(1+\frac{\beta^2}{2}x^2-\frac{\beta^4}{8}x^4+\frac{\beta^6}{16}x^6-\sqrt{1+\beta^2 x^2}\right).
\end{aligned} \tag{F.58}$$

Thus, we obtain the $(2n+1)$-dimensional solution

$$\Xi_\mu = \sum_{k=1}^{n} c_{2k}\tilde{\phi}_{2k}(x_\mu) + a_\mu\sqrt{1+\beta^2 x_\mu^2} - \frac{(-1)^n k^2}{x_\mu^2}\,. \tag{F.59}$$

The $(2n+1)$-dimensional solution contains parameters c_{2k} $(k = 1, \cdots, n)$, a_μ $(\mu = 1, \cdots, n)$, k^2 and β. In the limit $\beta \to 0$, we have $\phi_{2k} \to x^{2k}/2k$, which reproduces Kerr-NUT-(A)dS metrics in $2n+1$ dimensions [61]. We have

$$\frac{2\rho}{D-2} = -c_{2n} - 3\beta^2\tilde{P}_T\,,\ \frac{2p}{D-2} = c_{2n} + \beta^2\tilde{P}_T\,. \tag{F.60}$$

Hence, the equation of state is $\rho + 3p = (D-2)c_{2n}$ like the even dimensional case.

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